Valuation of Financial Options

Outline

- Examined options basics
  - Rights, not obligations
  - Asymmetric payoffs (limited losses)
  - Value different from immediate payoff
  - Value increases with volatility of underlying asset, time to expiration
  - Current stock price and option strike price also affect value

- What is exact option value?
  - Why NPV does not work for options
  - Boundaries on option value
  - Replicating option returns
  - The Black-Scholes model
  - A general binomial model

- Goal: build background for understanding real options
The Problem of Pricing Financial Options

- Traditional NPV not applicable
  NPV requires two steps
  Estimation of cash flows
  Discounting to present using risk-adjusted rate from CAPM

Step 1: option cash-flows (payoffs) depend on stock price
Future stock price is uncertain
Could describe with probability distribution

Step 2: adjusted discount rate depends on risk of option
Option risk changes with stock price
Stock prices change continually and unpredictably
Cannot predict option risk over time
No single, risk-adjusted, discount rate applies

Why Call Option Risk Changes Unpredictably

- Payout becomes more certain with increased S
  Possibility of losing entire investment decreases
  Decreases volatility (risk)

Risk of option changes every time stock price changes
Stock price changes continually and unpredictably
Moving Toward Options Pricing

- Need a framework other than NPV for valuing options
  - Still want to account for time value of money and risk
  - Begin by identifying logical constraints on price

Narrowing the Scope: Boundaries on Price

- Some logical boundaries on the price of an American call

  \[
  \text{Price} \geq 0 \\
  \text{Otherwise buy option immediately}
  \]

  \[
  \text{Price} \leq S \\
  \text{Stock yields } S^* \\
  \text{Option yields } S^* - K \\
  \text{Option worth less than stock}
  \]

  \[
  \text{Price} \geq S - K \\
  \text{Or buy and exercise immediately}
  \]
Valuation By Comparison

- Identified several influences and boundaries of options value

- Still do not have concrete option valuation method

- One idea is to replicate options payoffs using other assets
  - If end payoffs are the same, then
  - The initial value of these assets and the option should be equal

- Key is to find replicating assets that can be valued directly

Breaking a Call Option into Separate Components

- If exercised, call option results in stock ownership
  - Option owner effectively controls shares of stock

- Payment for stock delayed until option is exercised
  - Delayed payments are essentially loans

- Call options are like buying stock with borrowed money

- Use this analogy to develop estimate of option value
A One-Period Example

- **Stock**
  - Current price = $100
  - Price at end of period either $80 or $125

- **One-period call option**
  - Strike price = $110

- **Assume funds can be borrowed at risk-free rate**
  - One-period risk-free rate = 10%

- **Identify conditions where end-of-period payoffs are equal**
  - Buying stock and borrowing money
  - Buying call options

- **Then, initial values should be equal**

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Call Option Cost and Payoffs

- Pay C dollars to acquire option

- If S>K, call payoff = S - K

- If S<K, call payoff = 0

<table>
<thead>
<tr>
<th></th>
<th>Start (Stock = 100)</th>
<th>End (Stock = 80)</th>
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<tbody>
<tr>
<td>Buy Call</td>
<td>- C</td>
<td>0</td>
<td>(125 - 110) = 15</td>
</tr>
<tr>
<td>Strike = 110</td>
<td></td>
<td></td>
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Stock and Loan Cost and Payoffs

- Buy stock and borrow to have payoffs look like option
- If S>K, want stock and loan payment to net to positive return
  - Can develop ratio to equalize stock and loan payments to option returns
- If S<K, want stock and loan payment to net to zero

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Stock and Loan Cost and Payoffs (2)

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<tr>
<td>Buy Stock</td>
<td>-100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Borrow Money</td>
<td>80/(1+r)</td>
<td>-80</td>
<td>-80</td>
</tr>
<tr>
<td>Net</td>
<td>-100 + 80/(1+r)</td>
<td>0</td>
<td>45</td>
</tr>
</tbody>
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Comparing Costs and Payoffs

- If S>K, stock and borrowing returns more than call
  Ratio of returns in this case is 3:1

If S<K, returns are equal
Buying 3 calls should equalize payoffs

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<td>Buy Stock and Borrow</td>
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Equalizing Costs and Payoffs

- Equal payoffs suggest initial costs should be equal
  Otherwise could buy cheaper alternative and sell
  more expensive result would be instant profit

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<td>-3C</td>
<td>0</td>
<td>125 - 110 = 45</td>
</tr>
<tr>
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<td>Start (Stock = 100)</td>
<td>Start (Stock = 80)</td>
<td>End (Stock = 125)</td>
</tr>
<tr>
<td>Buy Stock and Borrow</td>
<td>100 + 80/(1.1)</td>
<td>0</td>
<td>45</td>
</tr>
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• 3C = -100 + 80/(1.1), therefore C = $9.09
One-Period Example Summary

- Call option payoff replicated using stock and borrowing
  - Cost of loan and price of stock are known
  - Allows value of option to be assessed

- Information needed to determine call value
  - Stock price
  - Strike price
  - Time (one-period)
  - Volatility of stock (range of final prices)
  - Interest rate also influenced option value

Options Pricing Models

- Concept of example important, must extend to be practical
  - Multiple periods
  - Dividends

- Present two options valuation frameworks

- Black-Scholes
  - Reasonably compact formula
  - Prices European calls only (assumes exercise can occur only at expiration)
  - Can be modified to include dividends

- A more general binomial model
  - Less limited in scope, possibly more difficult to apply
  - Considers exercise at any time and dividends
The Black-Scholes Options Pricing Formula

- The value of a European call on a non-dividend paying stock

\[ C = S \times N(d_1) - K \times e^{-rt} \times N(d_2) \]

- \( S \) = current stock price
- \( K \) = striking price
- \( r \) = risk-free rate of interest
- \( t \) = time to expiration
- \( \sigma \) = standard deviation of returns on stock
- \( N(x) \) = standard cumulative normal distribution

\[
d_1 = \frac{\ln \left( \frac{S}{K \times e^{-rt}} \right)}{\sigma \sqrt{t}} + (\sigma \sqrt{t}) \]

\[
d_2 = d_1 - (\sigma \sqrt{t}) \]

The Black-Scholes Options Pricing Formula (2)

- Note similarities to previous examples
  - Same factors required
  - Volatility replaces stock outcomes from one-period example
  - Resembles replicating portfolio (buy stock and borrow)

- Derivation complicated, not the focus here
Using Black-Scholes Model

- Essentially a substitution and solve formula
  - Programmed into most financial calculators
  - Ubiquitous to Wall Street community

- $S$, $K$, $t$ are directly stated terms of option

- $r$ is risk-free rate of currency named in strike price

- Volatility of stock must be estimated from historical data

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A Relationship Between Calls and Puts

- Put-Call parity
  - Put option value can be determined indirectly using Black-Scholes
  - For European options, on non-dividend paying stocks
    \[ C = P + S - Ke^{-rt} \]
Including Dividends in Black-Scholes

- Two adjustment methods

- Assumption of constant dividend yield
  Replace $S$ in formula with $S'(1-d)^n$
  $d$ = constant dividend yield
  $n$ = number of dividend periods

- Estimation of present value of dividends
  Replace $S$ in formula with $S-D$
  $D$ = present value of dividends

- Put-call parity becomes either
  \[
  C = P + S'(1-d)^n - Ke^{-rt}
  \]
  \[
  C = P + S - D - Ke^{-rt}
  \]

Limitations to Black-Scholes

- Black-Scholes values European options
  Most traded options are American type...
  As are most real options

- American options can be exercised at any time
  In general, early exercise is not optimal (option more valuable than payoff)
  Sometimes can be an extremely valuable feature

  Overall, a more general approach is needed
Example of Early Exercise Being Valuable

- Company unexpectedly decides to pay large dividend
  - Option lasts well beyond payment date
  - Stock will be worth much less after dividend payment, so will option
  - If option is in the money, would make sense to exercise just prior to pay-out

- Looking ahead to real options
  - Opportunity cost acts like dividends
  - Early exercise possible and likely

Origin of Black-Scholes Model

- One-period example
  - Compared end-of-period option value to stock and borrowing portfolio value
  - Equated beginning-of-period option value to initial portfolio value

- Black-Scholes model
  - Assumes many small periods
  - Represents limit as time period approaches zero
  - Calculates call option value based on statistically described stock movements
  - Assumes early exercise is not possible

- Needed for general model
  - Ability to decide to hold or exercise, at beginning of each period
A General Binomial Model for Options

- One-period call option example
  - Compared option value to portfolio of stock and borrowing
  - If stock price increased, call option had positive value
  - If stock price decreased, call option was worthless

- In reality, stock price continues to change over many periods
  \[ S \longrightarrow \begin{array}{l}
  S_u \\
  S_d \\
  S_{uu} \\
  S_{dd} \\
  S_{ud} \\
  \end{array} \]
  \[ u = \text{up} \quad d = \text{down} \]

- Option value changes each time stock price changes
  \[ C \longrightarrow \begin{array}{l}
  C_u \\
  C_d \\
  C_{uu} \\
  C_{dd} \\
  C_{ud} \\
  \end{array} \]

General Binomial Model Procedure

- Assumes many periods
- Works backward from date of expiration
- For each period, applies one-period valuation methodology
- At each node, compares
  - Value of option
  - Immediate exercise payoff
- Optimal policy determined for each period and stock price
  - Hold option for another period
  - Exercise immediately
General Binomial Model Results
(Single Period)

- Value of call if held for single period

\[ C = \frac{p \cdot Cu + (1-p) \cdot Cd}{1+r} \]

- where, \( p \) acts as a probability
- \( Cu \) and \( Cd \) determined by stock volatility

- Value of option is maximum of
  - Immediate exercise
  - Holding for another period
  - Zero

\[ C = \max\{S-K, \frac{p \cdot Cu + (1-p) \cdot Cd}{1+r}, 0\} \]

General Binomial Model Results
(Multi-period)

- Many periods are treated like a decision tree

  - Period 0
  - Period 1
  - Period 2
  - Period 3

  \[ C \]

- Work backward from last to first period to value \( C \)

- Apply one-period methodology at each node
  - example:
  \[ Cuu = \max\{Suu-K, \frac{p \cdot Cuuu + (1-p) \cdot Cuud}{1+r}, 0\} \]
Comments on Binomial Model

- Binomial model is a recursive technique
  - Start with end-period values and work backward to present
  - Tedious for anything other than short examples
  - Can be automated in computer programs

- Note similarity to NPV
  - Estimate cash-flows (end-of-period option value)
  - Discount to present (using risk-free rate)

\[ C = \frac{p \times Cu + (1-p) \times Cd}{(1+r)} \]

- Why does this work?
  - Started out on premise that NPV does not work for options
  - Model adjusts cash-flows such that risk-free rate is proper discount rate
  - Probability p is the mechanism for adjusting the cash-flows

Comparing Traditional NPV to Options Valuation

- Traditional NPV procedure
  - Estimate cash flows
  - Discount at risk-adjusted rate from CAPM

- Traditional NPV does not work for options

- Option valuation handles risk-adjustment differently
  - Estimate cash-flows and adjust for risk
  - Discount at risk-free rate

- Options procedure also known as risk neutral valuation
  - Critical concept of derivatives field
Summary

- Options cannot be valued using NPV
  - Risk constantly changes
  - Proper risk-adjusted discount rate cannot be determined

- Options valuation procedures use risk-neutral valuation
  - Adjust cash flows and apply risk-free rate
  - Versus adjust discount rate and apply to cash-flows

- Black-Scholes is compact, but limited
  - Values European calls
  - Put-call parity works for valuing puts

- Binomial model more general
  - A recursive technique
  - More complicated, but can be automated

Appendix: Observed Option Price Influences

- Combined list of influences
  - Underlying price (S)
  - Strike price (K)
  - Time to expiration (t)
  - Risk-free rate of interest (r)
  - Range (volatility) of stock price changes
  - Dividends (D)
  - American vs European options
  (ability to exercise early)
Appendix: Impact of Individual Factors on Option Value

<table>
<thead>
<tr>
<th>Factor/Option Type</th>
<th>American Call</th>
<th>American Put</th>
<th>European Call</th>
<th>European Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>Underlying Price</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Striking Price</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Time to Expiration</td>
<td>+</td>
<td>+</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>Volatility of Underlying</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Risk-free rate of interest</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Dividends</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

Appendix: Rationale Behind Influence Factors Table

- **Stock price**
  - The greater the stock price (S) relative to striking price (K), the more likely a call (put) will be in (out of) the money.

- **Striking price**
  - The greater the striking price (K) relative to stock price (S), the less likely a call (put) will be in (out of) the money.

- **Time to expiration**
  - For American options, an option with a longer term to expiration is the same as an option with a shorter term, plus additional time.
  - European options cannot be exercised until the expiration date, so the extra time could cause harm relative to the shorter term option.
Appendix: Rationale Behind Influence Factors Table (2)

- **Volatility of underlying stock**
  - Since options have a zero downside and a positive upside, increased volatility increases the likelihood of finishing in the money.

- **Risk-free rate**
  - The striking price is paid or received in the future, and its present value is reduced by increased interest rates.
  - For calls, the striking price is paid in the future.
  - For puts, the striking price is received in the future

- **Dividends**: Stock prices adjust downward for dividend payments. This reduces (increases) the likelihood a call (put) will finish in the money.