Production Functions

Outline

1. Definition

2. Technical Efficiency

3. Mathematical Representation

4. Characteristics

Production Function - Basic Model for Modeling Engineering Systems

- **Definition:**
  
  Represents technically efficient transform of physical resources \( X = (X_1, \ldots, X_n) \) into product or outputs \( Y \) (may be good or bad)

- **Example:**
  
  Use of aircraft, pilots, fuel (the \( X \) factors) to carry cargo, passengers and create pollution (the \( Y \))

- **Typical focus on 1-dimensional output**
Technical Efficiency

- A Process is Technically Efficient if it provides Maximum product from a given set of resources $X = X_1, ... X_n$

- Graph:

```
Output

Max

Feasible Region

Resource
```

Mathematical Representation - General

- Two Possibilities
- Deductive -- Economic
  - Standard economic analysis
  - Fit data to convenient equation
  - Advantage - ease of use
  - Disadvantage - poor accuracy
- Inductive -- Engineering
  - Create system model from knowledge of details
  - Advantage - accuracy
  - Disadvantage - careful technical analysis needed
Mathematical Representation - Deductive

- Standard Cobb-Douglas Production Fnc.
  \[ Y = a_0 \prod X_i^{a_i} = a_0 X_1^{a_1} \ldots X_n^{a_n} \]
  - Interpretation: ‘\( a_i \)’ are physically significant
  - Easy estimation by linear least squares
    \[ \log Y = a_0 + \sum a_i \log X_i \]
- Translog PF -- more recent, less common
  \[ \log Y = a_0 + \sum a_i \log X_i + \sum \sum a_{ij} \log X_i \log X_j \]
  - Allows for interactive effects
  - More subtle, more realistic

Mathematical Representation - Inductive

- “Engineering models” of PF
- Analytic expressions
  - Rarely applicable: manufacturing is inherently discontinuous
  - Exceptions: process exists in force field, for example transport in fluid, river
- Detailed simulation, Technical Cost Model
  - Generally applicable
  - Requires research, data, effort
  - Wave of future -- not yet standard practice
Cooling Time, Part Weight, and Cycle Time Correlation

\[ \text{Cyc} = 8.78 + 1.35 \times \text{Tool} + 0.0152 \times \text{Weight} \]

PF: Characteristics

- Isoquants
- Marginal Products
- Marginal Rates of Substitution
- Returns to Scale
- Convexity of Feasible Region
Characteristic: Isoquants

- Isoquant is the Locus (contour) of equal product on production function

- Graph:

Important Implication of Isoquants

- Many designs are technically efficient
  - All points on isoquant are technically efficient
  - no technical basis for choice among them
  - Example:
    * little land, much steel => tall building
    * more land, less steel => low building

- System Design depends on Economics
- Values are decisive
Characteristic: Marginal Products

- Marginal Product is the change in output as only one resource changes

\[ MP_i = \frac{\partial Y}{\partial X_i} \]

- Graph:

Diminishing Marginal Products

- Math:

\[ Y = a_0X_1^{a_1} \ldots X_i^{a_i} \ldots X_n^{a_n} \]
\[ \frac{\partial Y}{\partial X_i} = \frac{(a_i/X_i)Y}{(X_i^{a_i-1})} = f(X_i^{a_i-1}) \]

Diminishing Marginal Product if \( a_i < 1.0 \)

- “Law” of Diminishing Marginal Products
  - Commonly observed -- but not necessary
  - “Critical Mass” phenomenon => increasing marginal products
Characteristic: Marginal Rate of Substitution

- Marginal Rate of Substitution is the rate at which one resource must substitute for another so that product is constant.

- Graph:

\[
\begin{align*}
\Delta X_i & \quad \Delta X_j \\
X_i & \quad X_j
\end{align*}
\]

Marginal Rate of Substitution (cont’d)

- Math:
  \[
  \text{since } \Delta X_i MP_i + \Delta X_j MP_j = 0 \quad \text{(no change in product)}
  \]
  \[
  \text{then } MRS_{ij} = \frac{\Delta X_i}{\Delta X_j} = -\frac{MP_j}{MP_i} = -\left(\frac{a_j}{a_i}\right)\left(\frac{X_i}{X_j}\right)
  \]

- MRS is “slope” of isoquant
  - Note: It is negative
  - Loss in 1 dimension made up by gain in other
Characteristic: Returns to Scale

- Returns to Scale is the Ratio of rate of change in $Y$ to rate of change in ALL $X$ (each $X_i$ changes by same factor)

- Graph:
  - Directions in which the rate of change in output is measured for MP and RTS

Returns to Scale (cont’d)

- Math:
  
  $Y' = a_0 \pi X_i^{a_i}$
  
  $Y'' = a_0 \pi (sX_i)^{a_i}$
  
  $= Y'(s)^{\Sigma a_i}$

  
  $\text{RTS} = \frac{Y''/Y'}{s} = s^{(\Sigma a_i - 1)}$

  
  $Y''/Y' = \% \text{ increase in } Y$
  
  if $Y''/Y' > s \Rightarrow \text{Increasing RTS}$

  
  Increasing returns to scale if $\Sigma a_i > 1.0$
Importance of Increasing Returns to Scale

- Increasing RTS means that bigger units are more productive than small ones
- IRTS => concentration of production into larger units
- Examples:
  - Generation of Electric power
  - Chemical, pharmaceutical processes

Practical Occurrence of Increasing Returns to Scale

- Frequent!
- Generally where
  - Product = f(volume) and
  - Resources = f(surface)
- Example:
  - ships, aircraft, rockets
  - pipelines, cables
  - chemical plants
  - etc.
Characteristic: Convexity of Feasible Region

- A region is convex if it has no “reentrant” corners

- Graph:

  ![Convex and Not Convex Regions]

Test for Convexity of Feasible Region (cont’d)

- Math: If A, B are two vectors to any 2 points in region

  Convex if all
  \[ T = KA + (1-K)B \quad 0 \leq K \leq 1 \]
  entirely in region

  ![Convex Region Diagram]
Convexity of Feasible Region for Production Function

- Feasible region of Production function is convex if no reentrant corners

Convex

Non-Convex

- Convexity => Easier Optimization
  - by linear programming (discussed later)

Test for Convexity of Feasible Region of Production Function

- Test for Convexity: Given A,B on PF
  If \( T = KA + (1-K)B \) \( 0 \leq K \leq 1 \)
  Convex if all \( T \) in region

Cobb-Douglas: \( a_i \leq 1.0 \) and \( \sum a_i \leq 1.0 \)