Marginal Analysis Outline

1. Definition and Assumptions
2. Optimality criteria
   - Analysis
   - Interpretation
   - Application
3. Key concepts
   - Expansion path
   - Cost function
   - Economies of scale
4. Summary

Marginal Analysis

- Basic form of optimization of design
- Combines:
  - Production function - Technical efficiency
  - Input cost function, \( c(X) \)
  - Economic efficiency
Assumptions of Marginal Analysis

- Feasible region is convex (over relevant portion) This is key. Why?
- To guarantee no other optimum missed
- No constraints on resources
- Models are “analytic” (needed for proof)
- Defines optimum by looking at the margins -- the derivatives

Optimality Conditions for Design, by Marginal Analysis

The Problem:
\[
\begin{align*}
\text{Min } C(Y') &= c(X) & \text{cost of inputs} \\
\text{s.t. } g(X) &= Y' & \text{production function}
\end{align*}
\]

The Lagrangean:
\[
L = c(X) - \lambda [g(X) - Y']
\]
Optimality Conditions for Design, by Marginal Analysis (2)

- Key Result:
  \[ \frac{\partial c(X)}{\partial X_i} = \lambda \frac{\partial g(X)}{\partial X_i} \]
  \( \lambda \) is the marginal product (MP) and \( \lambda \) is the marginal cost (MC).

- Optimality Conditions:
  \[ \frac{MP_i}{MC_i} = \frac{MP_j}{MC_j} = \frac{1}{\lambda} \]
  \[ \frac{MC_j}{MP_j} = \frac{MC_i}{MP_i} = \lambda = \text{Shadow Price on Product} \]

- A balanced design,
  Each \( X_i \) contributes “same bang for buck”

Graphical Interpretation of Optimality Conditions

(A) Input Cost Function

\[ c(X) = \sum p_i X_i \leq B \]

- Linear case: In general, non-linear (as in curved line)

- B = Budget

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Graphical Interpretation of Optimality Conditions (2)

(B) Conditions

\[ \text{Slope} = MRS_{ij} = - \frac{MP_1}{MP_2} = - \frac{MC_1}{MC_2} \]

Application of Optimality Conditions

Problem: \( Y = a_0 X_1^{a_1} X_2^{a_2} \)
\[ c(X) = \sum p_i X_i \]

Note: Linearity of Input Cost Function
- typically assumed by economists
- in general, not valid
  - prices rise with demand
  - wholesale, volume discounts

Solution:
\[ \left[ \frac{a_1}{X_1^*} \right] Y / p_1 = \left[ \frac{a_2}{X_2^*} \right] Y / p_2 \]
\[ \text{(* denotes an optimum value)} \]
Expansion Path

- Locus of all optimal designs $X^*$
- Not a property of technical system alone
- Depends on local prices
- Optimal designs do not, in general, maintain constant ratios between optimal $X_i^*$
  e.g.: crew of 20,000 ton ship
  crew of 200,000 ton ship

Calculation of Expansion Path

- Assume: $Y = 2X_1^{0.48}X_2^{0.72}$ (increasing RTS)
  $c(X) = X_1 + X_2^{1.5}$ (increasing costs)
- Optimality Conditions:
  $\frac{0.48}{X_1} \frac{Y}{1} = \frac{0.72}{X_2} \frac{Y}{1.5X_2^{0.5}}$
  $= \frac{MP_1}{MC_1}$
  $\Rightarrow X_1^* = (X_2^*)^{1.5}$
- Graphically:
Cost Functions

- Not same as input cost function
  It represents the optimal cost of Y
  Not the cost of any set of $X$

- $C(Y) = C(X^*) = f(Y)$

Cost Functions (2)

- Graphically:

  ![Graphical representation of cost functions]

  - Feasible
  - Economists view
  - Engineers view
  - Cost-effectiveness

- Great practical use:
  How much Y for budget?
  $\Delta Y$ for $\Delta B$?
  Cost effectiveness, $\Delta B / \Delta Y$
Calculation of Cost Function

- Cobb-Douglas Prod. Fcn: \( Y = a_0 \pi X_i^{a_i} \)
- Linear input cost function: \( c(X) = \sum p_i X_i \)
- Result
  \[ C(Y) = A(\pi p_i^{a_i/r})Y^{1/r} \quad \text{where} \quad r = \sum a_i \]
- Easy to estimate statistically
  \( \Rightarrow \) Solution for ‘\( a_i \)’
  \( \Rightarrow \) Estimate of prod. fcn. \( Y = a_0 \pi X_i^{a_i} \)

Calculation of Cost Function (2)

- Assume Again:
  \( Y = 2X_1^{0.48} X_2^{0.72} \)
  \( c(X) = X_1 + X_2^{1.5} \)
- Expansion Path: \( X_1^* = (X_2^*)^{1.5} \)
  Thus: \( Y = 2(X_2^*)^{1.44} \)
  \( c(X^*) = 2(X_2^*)^{1.5} \)
  \( \Rightarrow \)
  \( X_2^* = (Y/2)^{0.7} \)
  \( c(Y) = c(X^*) = (2^{-0.05})Y^{1.05} \)
Economies of Scale

- A possible characteristic of cost function

- Concept similar to returns to scale, except
  - ratio of ‘X_i’ not constant
  - refers to costs (economies) not “returns”

- Economies of scale exist if costs increase slower than product
  \[ \text{Total cost} = C(Y) = Y^\alpha \quad \alpha < 1.0 \]

Economies of Scale (2)

- Note:
  - If Cobb-Douglas, linear input costs
  - Increasing RTS
    \[ r = \sum a_i > 1.0 \quad \Rightarrow \quad C(Y) = \text{fcn } Y^{1/r} \]

  Not necessarily true in general
  - See example!!
  \[ c(Y) = c(X^*) = (2^{-0.05})Y^{1.05} \]
Marginal Analysis Summary

- Assumptions --
  -- convex feasible region
  -- Unconstrained

- Optimality Criteria
  - MC/MP same for all inputs

- Expansion path -- Locus of Optimal Design

- Cost function -- cost along Expansion Path

- Economies of scale (vs Returns to Scale)
  -- Exist if Cost/Unit decreases with volume