Production Functions (PF)

Outline
1. Motivation
2. Definition
3. Technical Efficiency
4. Mathematical Representation
5. Characteristics

Production Function - Motivation

- In order to analyze a system, we need to model it, that is, provide connection between what we do, and what results
- Moreover, we need to focus our attention on the most interesting possibilities…
- This is role of “Production Function”
- It is basic Conceptual Structure for Modeling Engineering Systems
Production Function - Definition

- **Definition:**
  - Represents technically efficient transform of physical resources $X = (X_1, \ldots, X_n)$ into product or outputs $Y$ (may be good or bad)

- **Example:**
  - Use of aircraft, pilots, fuel (the $X$ factors) to carry cargo, passengers and create pollution (the $Y$)

- **Typical focus on 1-dimensional output**

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Technical Efficiency

- **A Process is Technically Efficient if it provides Maximum product from a given set of resources $X = X_1, \ldots, X_n$**

- **Graph:**
  ![Graph](image)
Mathematical Representation -- General

- Two Possibilities
  - Deductive -- Economic
    - Standard economic analysis
    - Fit data to convenient equation
    - Advantage - ease of use
    - Disadvantage - poor accuracy
  - Inductive -- Engineering
    - Create system model from knowledge of details
    - Advantage - accuracy
    - Disadvantage - careful technical analysis needed

Mathematical Representation -- Deductive

- Standard Cobb-Douglas Production Function $Y$
  \[ Y = a_0 \prod X_i^{a_i} = a_0 X_1^{a_1} ... X_n^{a_n} \]
  - $\pi$ means multiplication
  - Interpretation: ‘$a_i$’ are physically significant
  - Easy estimation by linear least squares
    \[ \log Y = \log a_0 + \sum a_i \log X_i \]
- Translog PF -- more recent, less common
  \[ \log Y = a_0 + \sum a_i \log X_i + \sum \sum a_{ij} \log X_i \log X_j \]
  - Allows for interactive effects
  - More subtle, more realistic
- Economist models (no technical knowledge)
PF Example

- One of the advantages of the “economist” models is that they make calculations easy. This is good for examples, even if not as realistic as Technical Cost Models (next)

- Thus: \( \text{Output} = 2 \times M^{0.4} \times N^{0.8} \)

- Let’s see what this looks like...

PF Example -- Calculation

<table>
<thead>
<tr>
<th>M</th>
<th>N</th>
<th>Output</th>
<th>N VARIABLE</th>
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<td>10</td>
<td>31.70</td>
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<td>72.82</td>
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<td>49.19</td>
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<td>60.34</td>
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<td>105.06</td>
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</tbody>
</table>

The formula in Excel to calculate the output is: \( = 2 \times (\text{power(b7,0.4)}) \times (\text{power(c7,0.8)}) \)

We calculate output for many values of the variables using a 2-way Data Table

Recall: \( \text{Output} = 2 \times M^{0.4} \times N^{0.8} \)
PF Example -- Graphs

[Diagram showing 3D graph with axes labeled as OUTPUT, M VARIABLE, and N VARIABLE.]

PF Example -- Graphs

[Diagram showing 3D graph with axes labeled as OUTPUT, M VARIABLE, and N VARIABLE.]

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Mathematical Representation -- Inductive

- “Engineering models” of PF
- Analytic expressions
  - Rarely applicable: manufacturing is inherently discontinuous
  - Exceptions: process exists in force field, for example transport in fluid, river
- Detailed simulation, Technical Cost Model
  - Generally applicable
  - Requires research, data, effort
  - Wave of future -- not yet standard practice

Cooling Time, Part Weight, Cycle Time Correlation (MIT MSL, Dr. Field)

Regression Curve-Fit
\[ T_{\text{erc}} = 8.78 + 1.35 \times T_{\text{ool}} + 0.0152 \times \text{Weight} \]
PF: Characteristics

- Isoquants
- Marginal Products
- Marginal Rates of Substitution
- Returns to Scale
- Possible Convexity of Feasible Region

Characteristic: Isoquants

- Isoquant is the Locus (contour) of equal product on production function

- Graph:
Important Implication of Isoquants

- Many designs are technically efficient
  - All points on isoquant are technically efficient
  - no technical basis for choice among them
  - Example:
    * little land, much steel => tall building
    * more land, less steel => low building

- Best System Design depends on Economics
- Values are decisive!

Isoquant Example -- Calculation

For any given output, we can calculate the M value as a function of the N value. Thus:
for output = 20, the formula is:

\[ M = \text{power}(10, 2.50 / (\text{power}(c7, 2)) \]

A 1-way data table calculates the (M,N) combinations that constitute the isoquant

<table>
<thead>
<tr>
<th>M for OUTPUT= 20</th>
<th>c7=10</th>
<th>3.16</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
<td>35.14</td>
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<td>11</td>
<td>2.61</td>
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<td>13</td>
<td>1.87</td>
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<td>15</td>
<td>1.41</td>
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</tbody>
</table>

Recall: Output = 2 M 0.4 N 0.8
Isoquant Example -- Graph

Characteristic: Marginal Products

- Marginal Product is the change in output as only one resource changes
  \[ MP_i = \frac{\partial Y}{\partial X_i} \]
  - Graph:
    \[ MP_i \]
    \[ X_i \]
Diminishing Marginal Products

- Math:
  \[ Y = a_0X_1^{a_1} \cdots X_i^{a_i} \cdots X_n^{a_n} \]
  \[ \frac{\partial Y}{\partial X_i} = \left(\frac{a_i}{X_i}\right)Y = f(X_i^{a_i-1}) \]
  Diminishing Marginal Product if \( a_i < 1.0 \)

- “Law” of Diminishing Marginal Products
  - Commonly observed -- but not necessary
  - “Critical Mass” phenomenon => increasing marginal products

MP Example -- Calculations

<table>
<thead>
<tr>
<th>C7=10</th>
<th>1.53</th>
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<tbody>
<tr>
<td>3</td>
<td>3.15</td>
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<td>5</td>
<td>2.32</td>
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<td>1.20</td>
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The formula for the marginal product is

\[ = (0.4/b7)*(2)*(power(b7,0.4))*power(12.65,0.8) \]

Note that the Marginal Product is conditional on the change in only one variable (in this case M). All other variables are fixed (in this case N=12.65).

Obviously, the Marginal Product depends on the “cut” of the production function you take.

Recall: \( \text{Output} = 2M^{0.4}N^{0.8} \)
MP Example -- Graph

Characteristic: Marginal Rate of Substitution

- Marginal Rate of Substitution is the Rate at which one resource must substitute for another so that product is constant

- Graph:

\[ X_j \]

\[ \Delta X_i \]

\[ \Delta X_j \]

Isoquant
Marginal Rate of Substitution (cont’d)

- Math:
  
  since \( \Delta X_I M_P + \Delta X_J M_P = 0 \)
  
  (no change in product)
  
  then \( MRS_{IJ} = \frac{\Delta X_J}{\Delta X_I} \)
  
  \[
  = \frac{-MP_I}{MP_J} = -\left[\frac{a_I}{X_I}\right]/\left[\frac{a_J}{X_J}\right]
  \]
  
  \[
  = -\left(\frac{a_i}{a_j}\right)\left(\frac{X_J}{X_I}\right)
  \]
  
- \( MRS \) is “slope” of isoquant
  
  - It is negative
  
  - Loss in 1 dimension made up by gain in other

MRS Example

- For our example PF: Output = 2 M^{0.4} N^{0.8}
  
  \( a_M = 0.4 \); \( a_N = 0.8 \)
  
  At a specific point, say \( M = 5, N = 12.65 \)
  
  \( MRS = -\left(\frac{0.4}{0.8}\right)\left(\frac{12.65}{5}\right) = -1.265 \)
  
  At that point, it takes ~ 5/4 times as much M as N to get the same change in output
Characteristic: Returns to Scale

- Returns to Scale is the Ratio of rate of change in Y to rate of change in ALL X (each $X_i$ changes by same factor)

- Graph:
  - Directions in which the rate of change in output is measured for MP and RTS

$X_j$  

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Returns to Scale (cont’d)

- Math:
  $Y' = a_0 \prod X_i a_i$
  $Y'' = a_0 \prod (sX_i)^a_i = Y'(s)^{\Sigma a}$  all inputs increase by s

$\text{RTS} = (Y''/Y')/s = s^{(\Sigma a_i - 1)}$

$Y''/Y' = \% \text{ increase in Y}$

if $Y''/Y' > s \Rightarrow \text{Increasing RTS}$

Increasing returns to scale (IRTS) if $\Sigma a_i > 1.0$
Increasing RTS Example

- The PF is: $\text{Output} = 2 \ M^{0.4} \ N^{0.8}$
  - Thus $\sum a_i = 0.4 + 0.8 = 1.2 > 1.0$
  - So the PF has Increasing Returns to Scale
  - Compare outputs for (5,10), (10,20), (20,40)

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<tr>
<th>N VARIABLE</th>
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Importance of Increasing RTS

- Increasing RTS means that bigger units are more productive than small ones

- IRTS => concentration of production into larger units

- Examples:
  - Generation of Electric power
  - Chemical, pharmaceutical processes
Practical Occurrence of IRTS

- Frequent!
- Generally where
  - Product = \( f(\text{volume}) \) and
  - Resources = \( f(\text{surface}) \)
- Example:
  - ships, aircraft, rockets
  - pipelines, cables
  - chemical plants
  - etc.

Characteristic: Convexity of Feasible Region

- A region is convex if it has no “reentrant” corners

- Graph:
  - CONVEX
  - NOT CONVEX
Informal Test for Convexity of Feasible Region (cont’d)

- Math: If A, B are two vectors to any 2 points in region

  Convex if all
  \[ T = KA + (1-K)B \quad 0 \leq K \leq 1 \]
  entirely in region

Convexity of Feasible Region for Production Function

- Feasible region of Production function is convex if no reentrant corners

- Convexity => Easier Optimization
  - by linear programming (discussed later)
Test for Convexity of Feasible Region of Production Function

- Test for Convexity: Given A,B on PF
  If \( T = KA + (1-K)B \quad 0 \leq K \leq 1 \)
  Convex if all \( T \) in region

- For Cobb-Douglas, the test is if:
  all \( a_i \leq 1.0 \) and \( \Sigma a_i \leq 1.0 \)

Convexity Test Example

- Example PF has Diminishing MP, so in the MP direction it looks like left side
- But: it has IRTS, like bottom of right side
- Feasible Region is not convex
Summary

- Production models are the way to describe technically efficient systems

- Important characteristics
  - Isoquants, Marginal products, Marginal rates of Substitution, Returns to scale, possible convexity

- Two ways to represent
  - Economist formulas
  - Technical models (generally more accurate)