Valuation of Financial Options (1)

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Outline

- Developing an intuitive sense of value
  - Asymmetric payoffs (limited losses)
  - Value different from immediate payoff
  - Value increases with: volatility of asset, time to expiration
  - Current stock price and option strike price also affect value

- Motivating Example of Value
  - Replicating outcomes
  => Value Independent of Objective probabilities!!!

- Arbitrage
  - Arbitrage Enforced Pricing as key concept
Definitions of Key Features

- $S$ = price (of stock, commodity, etc) at any time
- $S^*$ = price at time you exercise option
- $K$ = strike price at which item can be bought (call) or sold (put)
- $t$ = time remaining until option expires
- $\beta$ = standard deviation of returns (volatility)
- $r$ = risk-free rate of interest

Financial Options: Payoff

- Payoff = amount received from exercise of option

  Call Option Payoff
  - If exercised, option owner buys stock for a set price
    - Gets stock worth $S^*$ dollars
    - Pays strike price of $K$ dollars
    - Net position = $S^* - K$
  - If unexercised, net payoff is zero

  Net Payoff of Call Option:
  - Maximum of either 0 or $S^* - K$ = net payoff for call
  - Expressed as: $\text{Max}[0, S^* - K]$
  - This is ASYMMETRIC
Financial Options: Value

- Value often exceeds Payoff
  - Because variability of stock price can increase payoff of option
  - There is thus an expectation of greater value than immediate payoff

- Calculation of Value
  - requires sophisticated analysis
  - Determination of method for calculating value of options won Nobel Prize
  - Subject of Next Lectures

Sources of Value in Options

- Need to build up to valuation
  - Identify interesting features
  - Examine influences of value
  - Combine findings into valuation framework

- Start by looking at payoffs from options
  - Payoff structure influences on value
  - Generally, payoff and value of options are different

- Calls and then Puts
Payoff Diagram for Call Option

Valuation of Options

- How much should you pay to acquire an option?

- Payoff diagrams show for a given strike price
  - Call payoff increases with asset price
  - Put payoff decreases with asset price

- Immediate payoff generally does not reflect full value of option
  - Owner exercises only when advantageous
  - Must compare immediate exercise value with waiting
**Boundaries on Price**

Some logical boundaries on the price of an American call:

- Price > 0
  - Otherwise buy option immediately

- Price ≤ S
  - Stock yields S^∗
  - Option yields S^∗ - K
  - Option worth less than stock

- Price > S - K
  - Or buy and exercise immediately

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**Why Immediate Payoff and Value Differ**

- Consider an option “at the money”, that is, current price of asset equals strike price: (S = K)
  - Immediate exercise payoff is zero

However, if you wait:
- You might have higher payoff
- Worst is zero payoff (same as immediate exercise)

Value of waiting not reflected in immediate exercise

\[ \circ = \text{value of option} \]
Examining Value for All Stock Prices

- Value exceeds immediate exercise payoff
- Asymptotically approaches payoff for increased $S$
- If asset has no expectations: $value = 0 = value$ of option

Option Value Increases with Volatility

- Two at the money options ($S=K$) on different assets
  - Both options have average around payoff = 0
  - Asset A with greater volatility has more opportunity for large net payoffs: so it will have higher expected value

Asymmetric returns favor high variation (limited losses)
Impact of Time

- Increasing time to expiration increases value of American option (those you can exercise any time)
  - Ability to wait allows option owner to benefit from asymmetric returns
  - Longer-term contains shorter-term options, plus more time, must be better

- Compare a 3 and 6 month American call
  - Can exercise 6 month call at same time as 3 month
  - Can wait longer with 6 month
  - Which is more valuable? Must be longer one...

- Time impact less clear for European options
  - Forced to wait to exercise
  - Could miss out on profitable opportunities

Generalized American Call Option Value

- For a set strike price, value increases with
  - Stock price increases
  - Volatility
  - Time

- Increased strike price
  - Reduces likelihood of payoffs
  - Reduces call option value
Put Option Payoff

- Recall: Put Option gives person right to sell an asset for a predetermined “strike” price, $K$
  - Will do so only when value of asset, $S$, is less than $K$

- If exercised, put option owner sells asset
  - Sells asset worth $S^*$ dollars
  - Receive strike price of $K$ dollars
  - Net position $= K - S^*$

- If unexercised, net payoff is zero

- Net payoff for put $= \max[0, K - S^*]$

Payoff Diagram for Put Option
Generalized American Put Option Value

- For a set strike price, put option value increases with
  - Stock price declines
  - Volatility
  - Time

- Increased strike price
  - Increases likelihood of payoffs
  - Increases put option value

Option Valuation: Motivating Example

- The valuation of this very simple option has fundamentally important lessons

- The key idea is that it is possible to replicate options payoffs using a portfolio of other assets
  - Since option and portfolio has same end payoffs, then
  - The value of option = value of portfolio

- Surprisingly, value of option does NOT depend on probability of payoffs!
  - Contrary to intuition associated with probabilistic nature
  - This surprising insight is basis for options analysis

  => Arbitrage enforced valuation (to be defined)
A Simple 1 Period Option

- Asset has a Current price, \( S = $100 \)
- Price at end of period either $80 or $125
- One-period call option to buy asset at Strike price, \( K = $110 \)
- What is the value of this option?
- More precisely, what is the price, \( C \), that we should pay for this option?

Portfolio of Equal Outcomes

- The valuation is based on the idea that we can construct another asset that will have same payoffs
- This is a “replicating portfolio”
- By construction, the set of assets in portfolio will counter-balance each other to give same payoffs as the option
- Therefore: value of option = value of portfolio
- What does this portfolio look like?
Components of Replicating Portfolio

- Think about what a call option provides: It enables owner to get increased value of asset
  - If exercised, call option results in stock ownership

- However, options provides these benefits without much money! Payment for asset delayed until option is exercised
  - Ability to delay payment is equivalent to a loan

- Therefore: A Call option is like buying stock with borrowed money

- This analogy is basis for “Replicating Portfolio”

Call Option Cost and Payoffs

- Fair Cost of Option, C, is its value. This is what we want to determine

- If end of period asset price, S > K, strike price: payoff of the option = S - K
  If end of period asset price S < K, strike price: payoff of the option = 0

<table>
<thead>
<tr>
<th></th>
<th>Start</th>
<th>End</th>
<th>End</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset Price</td>
<td>100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Buy Call Strike = 110</td>
<td>- C</td>
<td>0</td>
<td>(125 - 110) = 15</td>
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</table>
Replicating Portfolio Cost and Payoffs

- Replicating Portfolio consists of:
  Asset bought at beginning of period
  Financed in part by borrowed money

- Amounts of Stock bought and money borrowed arranged so that payoffs equal those of option

- If $S > K$, want stock and loan payment to net to positive return

- If $S < K$, want stock and loan payment to net to zero
  Note: This is first crucial point of arrangement!

Table of Portfolio Cost and Payoffs

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<td>Asset price</td>
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<td>125</td>
</tr>
<tr>
<td>Buy Stock</td>
<td>-100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Borrow Money</td>
<td>80/(1+r)</td>
<td>- 80</td>
<td>- 80</td>
</tr>
<tr>
<td>Net</td>
<td>-100 + 80/(1+r)</td>
<td>0</td>
<td>45</td>
</tr>
</tbody>
</table>

Important Issue: What discount rate, $r$, should be used?
Comparing Costs and Payoffs of Option and Replicating Portfolio

- If $S < K$, both payoffs = 0 and are automatically equal.

- If $S > K$, portfolio payoff is a multiple of call payoff (in this case, ratio is 3:1)

- Thus, payoff of this multiple of calls (3) = portfolio payoff

<table>
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<td>125</td>
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<td>Buy Call</td>
<td>- C</td>
<td>0</td>
<td>(125-110) = 15</td>
</tr>
<tr>
<td>Buy Asset</td>
<td>-100 + 80/(1 + r)</td>
<td>0</td>
<td>45</td>
</tr>
<tr>
<td>And Borrow</td>
<td></td>
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Implications of: Option = Portfolio

- The obvious result is that if payoffs of having Option or Portfolio are equal, their values are equal

- Most important, however, is fact that person selling option can counter-balance this with a portfolio of equal value, so seller of option cannot lose!

- Note: Since above transaction has no risk, appropriate DISCOUNT RATE = RISK FREE RATE!

- Such a no-risk situation is known as ARBITRAGE
Arbitrage - Enforced Pricing

- The possibility of setting up a risk-free portfolio to balance option absolutely defines prices of option.

- This is known as “Arbitrage - Enforced Pricing”.

- If you pay \( C^* \) for option, where \( C^* > C \) (price defined by portfolio), than someone could sell you options and be sure to make money -- until you lower price to \( C \).

- Conversely, if you sell option for \( C^* < C \), then someone could buy them and make money until price = \( C \).

- Thus: \( C \) is price that must prevail, as calculated using risk-free discount rate!! THIS IS CRUCIAL INSIGHT!!

Value of Option

- Value of Option = Value of Portfolio

- This is easy to define, using risk-free discount rate, \( R_f \)
  - Calculation below assumes \( R_f = 10\% \) (for easy calculation)

\[
C = \frac{1}{3} \left[ -100 + \frac{80}{1 + R_f} \right] = \$ 9.09
\]

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</tr>
</thead>
<tbody>
<tr>
<td>Asset Price</td>
<td>100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Buy 3 Calls</td>
<td>-3C</td>
<td>0</td>
<td>45</td>
</tr>
<tr>
<td>Buy Asset</td>
<td>(-100 + 80/(1 + r))</td>
<td>0</td>
<td>45</td>
</tr>
<tr>
<td>And Borrow</td>
<td></td>
<td></td>
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Value independent of actual probability!

- Note Carefully!!

- Nowhere in the calculation of the option value is there any statement about actual probability that high or low payoffs (125 or 80) would occur.

- With arbitrage-enforced pricing, actual probabilities do not matter!

- This is a remarkable result. It is counter-intuitive. Since options deal with risks, it is very surprising.

- What does matter is the RANGE of possible payoffs.

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Summary

- Value of options INCREASES WITH RISK

- Value of Options is determined by Arbitrage -- This is the fundamental observation

- Arbitrage creates portfolio to match option payoffs -- being risk-free, evaluated at risk-free rate