Valuation of Financial Options

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Outline

- Developing an intuitive sense of value
  - Asymmetric payoffs (limited losses)
  - Value different from immediate payoff
  - Value increases with: volatility of asset, time to expiration
  - Current stock price and option strike price also affect value

- Motivating Example of Value
  - Replicating outcomes
  => Value independent of Objective probabilities!!!

- Arbitrage
  - Arbitrage Enforced Pricing as key concept

- Calculating option value
  - The Black-Scholes model
  - Binomial model
Sources of Value in Options

- Need to build up to valuation
  - Identify interesting features
  - Examine influences of value
  - Combine findings into valuation framework

- Start by looking at payoffs from options
  - Payoff structure influences on value
  - Generally, payoff and value of options are different

- Calls and then Puts

Call Option Payoff

- Recall: Call Option gives person right to buy an asset for a predetermined “strike” price, $K$
  - Will do so only when value of asset, $S$, is greater than $K$

- If exercised, call option owner buys asset
  - Gets asset worth $S^*$ dollars
  - Pays strike price of $K$ dollars
  - Net position $= S^* - K$

- If unexercised, net payoff is zero

- Net payoff for call is Maximum of either 0 or $S^* - K$

- Net payoff for call $= \text{Max} \ [0, S^* - K]$
Payoff Diagram for Call Option

<table>
<thead>
<tr>
<th>Payoff Diagram for Call Option</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff ($)</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>S</td>
</tr>
<tr>
<td>S - K</td>
</tr>
<tr>
<td>Price of Asset</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>Asset Price ($)</td>
</tr>
</tbody>
</table>

Valuation of Options

- How much should you pay to acquire an option?
- Payoff diagrams show for a given strike price
  - Call payoff increases with asset price
  - Put payoff decreases with asset price
- Immediate payoff generally does not reflect full value of option
  - Owner exercises only when advantageous
  - Must compare immediate exercise value with waiting
Some logical boundaries on the price of an American call

\[
\begin{align*}
\text{Price} & \geq 0 \\
\text{Otherwise buy option immediately}
\end{align*}
\]

\[
\begin{align*}
\text{Price} & \leq S \\
\text{Stock yields } S^* \\
\text{Option yields } S^* - K \\
\text{Option worth less than stock}
\end{align*}
\]

\[
\begin{align*}
\text{Price} & \geq S - K \\
\text{Or buy and exercise immediately}
\end{align*}
\]

Why Immediate Payoff and Value Differ

- Consider an option “at the money”, that is, current price of asset equals strike price: \( S = K \)
- Immediate exercise payoff is zero

However, if you wait:
- You might have higher payoff
- Worst is zero payoff (same as immediate exercise)

Value of waiting not reflected in immediate exercise

\[
\begin{align*}
\text{EV} & = \text{value of option}
\end{align*}
\]
Examining Value for All Stock Prices

- Value exceeds immediate exercise payoff
- Asymptotically approaches payoff for increased S
- If asset has no expectations: value = 0 = value of option

Option Value Increases with Volatility

- Two at the money options (S=K) on different assets
- Both options have average around payoff = 0
- Asset A with greater volatility has more opportunity for large net payoffs: so it will have higher expected value

Asymmetric returns favor high variation (limited losses)
Impact of Time

- Increasing time to expiration increases value of American option (those you can exercise any time)
  - Ability to wait allows option owner to benefit from asymmetric returns
  - Longer-term contains shorter-term options, plus more time, must be better

- Compare a 3 and 6 month American call
  - Can exercise 6 month call at same time as 3 month
  - Can wait longer with 6 month
  - Which is more valuable? Must be longer one...

- Time impact less clear for European options
  - Forced to wait to exercise
  - Could miss out on profitable opportunities

Generalized American Call Option Value

- For a set strike price, value increases with
  - Stock price increases
  - Volatility
  - Time

- Increased strike price
  - Reduces likelihood of payoffs
  - Reduces call option value

![Payoff vs Asset Price](image)
Put Option Payoff

- Recall: Put Option gives person right to sell an asset for a predetermined “strike” price, K
  - Will do so only when value of asset, S, is less than K

- If exercised, put option owner sells asset
  - Sells asset worth $S^*$ dollars
  - Receive strike price of $K$ dollars
  - Net position = $K - S^*$

- If unexercised, net payoff is zero

- Net payoff for put = Max [0, K - S*]

Payoff Diagram for Put Option
**Generalized American Put Option Value**

- For a set strike price, put option value increases with
  - Stock price declines
  - Volatility
  - Time

- Increased strike price
  - Increases likelihood of payoffs
  - Increases put option value

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**Option Valuation: Motivating Example**

- The valuation of this very simple option has fundamentally important lessons

- The key idea is that it is possible to replicate options payoffs using a portfolio of other assets
  - Since option and portfolio has same end payoffs, then
  - The value of option = value of portfolio

- Surprisingly, value of option does NOT depend on probability of payoffs!
  - Contrary to intuition associated with probabilistic nature
  - This surprising insight is basis for options analysis

- \( \Rightarrow \) Arbitrage enforced valuation (to be defined)
A Simple 1-Period Option

- Asset has a Current price, \( S = \$100 \)
- Price at end of period either $80 or $125
- One-period call option to buy asset at Strike price, \( K = \$110 \)
- What is the value of this option?
- More precisely, what is the price, \( C \), that we should pay for this option?

Portfolio of Equal Outcomes

- The valuation is based on the idea that we can construct another asset that will have same payoffs
- This is a “replicating portfolio”
- By construction, the set of assets in portfolio will counter-balance each other to give same payoffs as the option
- Therefore: value of option = value of portfolio
- What does this portfolio look like?
Components of Replicating Portfolio

- Think about what a call option provides: It enables owner to get increased value of asset
  - If exercised, call option results in stock ownership

- However, options provide these benefits without much money! Payment for asset delayed until option is exercised
  - Ability to delay payment is equivalent to a loan

- Therefore: A Call option is like buying stock with borrowed money

- This analogy is basis for “Replicating Portfolio”

Call Option Cost and Payoffs

- Fair Cost of Option, C, is its value. This is what we want to determine

- If end of period asset price, $S > K$, strike price:
  - payoff of the option = $S - K$
- If end of period asset price $S < K$, strike price:
  - payoff of the option = 0

<table>
<thead>
<tr>
<th>Asset Price</th>
<th>Start</th>
<th>End</th>
<th>End</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy Call</td>
<td>- C</td>
<td>0</td>
<td>$(125 - 110) = 15$</td>
</tr>
</tbody>
</table>
Replicating Portfolio Cost and Payoffs

- Replicating Portfolio consists of:
  - Asset bought at beginning of period
  - Financed in part by borrowed money

- Amounts of Stock bought and money borrowed arranged so that payoffs equal those of option

- If $S > K$, want stock and loan payment to net to positive return

- If $S < K$, want stock and loan payment to net to zero
  Note: This is first crucial point of arrangement!

#### Table of Portfolio Cost and Payoffs

<table>
<thead>
<tr>
<th>Asset price</th>
<th>Start</th>
<th>End</th>
<th>End</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Buy Stock</td>
<td>-100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Borrow Money</td>
<td>80/(1+r)</td>
<td>- 80</td>
<td>- 80</td>
</tr>
<tr>
<td>Net</td>
<td>-100 + 80/(1+r)</td>
<td>0</td>
<td>45</td>
</tr>
</tbody>
</table>

Important Issue: What discount rate, $r$, should be used?
Comparing Costs and Payoffs of Option and Replicating Portfolio

- If \( S < K \), both payoffs = 0 and are automatically equal.

- If \( S > K \), portfolio payoff is a multiple of call payoff (in this case, ratio is 3:1)

- Thus, payoff of this multiple of calls (3) = portfolio payoff

<table>
<thead>
<tr>
<th>Period</th>
<th>Start</th>
<th>End</th>
<th>End</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset Price</td>
<td>100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Buy Call</td>
<td>- C</td>
<td>0</td>
<td>(125-110) = 15</td>
</tr>
<tr>
<td>Buy Asset And Borrow</td>
<td>-100 + 80/(1 + r)</td>
<td>0</td>
<td>45</td>
</tr>
</tbody>
</table>

Implications of: Option = Portfolio

- The obvious result is that if payoffs of having Option or Portfolio are equal, their values are equal

- Most important, however, is fact that person selling option can counter-balance this with a portfolio of equal value, so seller of option cannot lose!

- Note: Since above transaction has no risk, appropriate DISCOUNT RATE = RISK FREE RATE!

- Such a no-risk situation is known as ARBITRAGE
**Arbitrage - Enforced Pricing**

- The possibility of setting up a risk-free portfolio to balance option absolutely defines prices of option.

- This is known as “Arbitrage - Enforced Pricing”

- If you pay $C^*$ for option, where $C^* > C$ (price defined by portfolio), than someone could sell you options and be sure to make money -- until you lower price to $C$. 

- Conversely, if you sell option for $C^* < C$, then someone could buy them and make money until price = $C$

- Thus: $C$ is price that must prevail, as calculated using risk- free discount rate!! THIS IS CRUCIAL INSIGHT!!

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**Value of Option**

- Value of Option = Value of Portfolio
- This is easy to define, using risk-free discount rate, $R_f$
  - Calculation below assumes $R_f = 10\%$ (for easy calculation)
  - $C = (1/3)[ -100 + 80/ (1 + R_f)] = $9.09

<table>
<thead>
<tr>
<th>Period</th>
<th>Start</th>
<th>End</th>
<th>End</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset Price</td>
<td>100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Buy 3 Calls</td>
<td>-3C</td>
<td>0</td>
<td>45</td>
</tr>
<tr>
<td>Buy Asset And Borrow</td>
<td>-100 + 80/ (1 + r)</td>
<td>0</td>
<td>45</td>
</tr>
</tbody>
</table>
Value independent of actual probability!

- Note Carefully!!

- Nowhere in the calculation of the option value is there any statement about actual probability that high or low payoffs (125 or 80) would occur.

- With arbitrage-enforced pricing, actual probabilities do not matter!

- This is a remarkable result. It is counter-intuitive. Since options deal with risks, it is very surprising.

- What does matter is the RANGE of possible payoffs.

Options Pricing Models

- Two basic options valuation frameworks

  - Black-Scholes
    - Historical Interest -- seminal basis for Nobel Prize
    - Compact formula widely used
    - Prices European calls only (assumes exercise can occur only at expiration)
    - Emphasizes concept of option as buying asset with loan

  - Binomial model
    - Generally applicable
    - Extends "replicating portfolio" through many periods
    - By making periods small, can get to very detailed analysis
**Black-Scholes Options Pricing Formula**

Value of a European call on a *non-dividend paying* asset is

\[ C = S \cdot N(d_1) - [K \cdot e^{-rt} \cdot N(d_2)] \]

- **S, K** = current price, strike price of asset
- **r** = risk-free rate of interest
- **t** = time to expiration
- **\(\sigma\)** = standard deviation of returns on asset
- **\(N(x)\)** = standard cumulative normal distribution

\[
\begin{align*}
\text{d}_1 & = \frac{\ln \left( \frac{S}{K} \right) + (r + 0.5 \sigma^2) t}{\sigma \sqrt{t}} \\
\text{d}_2 & = \text{d}_1 - \sigma \sqrt{t}
\end{align*}
\]

Formula like example: \( C = f(\text{asset price}) - g(\text{loan}) \) where standard deviation gives measure of range of payoffs

**Black-Scholes Background**

- **Basis of Model**
  - Assumes many small periods
  - Represents limit as time period approaches zero
  - Assumes early exercise is not possible

- **Calculates option value based on statistically described movements of value of asset**
  - These movements are supposed to be random motion around a trend
  - This is generally referred to as a Brownian motion or a Wiener process
  - More specifically, the motion is around logarithmic (multiplicative) scale so that values never negative
  - Empirical evidence supports these assumptions
Binomial Model of Valuation

- Binomial Model approximates movement of price of asset as a sequence of increases and decreases.

- Accuracy depends on number of stages of ups and downs allowed -- can be very detailed and accurate.

- As usual in options analysis, changes up and down assumed to be multiplications of previous values:
  - Thus, value of asset at beginning of period = S
  - Value at end if “up” = uS
  - Value at end if “down” = dS
  - Where “u” and “d” are parameters to be determined.

- Start with single period model, then extend.

Single Period Binomial Model Set-up

- Apply to generalized form of motivational example.

Value of Asset is up or down:

- S → uS
- S → dS

Value of call option is correspondingly up or down:

- C → Max(Su - K, 0) = Cu
- C → Max(Sd - K, 0) = Cd

Value of loan rises by Rf (no risk) to:

- R = 1 + Rf (for 1 year)
**Single Period Binomial Model Solution**

- The issue is to find what proportion of asset and loan to have to establish replicating portfolio

- If asset share is “x” and loan share is “y”, we solve:
  \[ xuS + yR = Cu \quad \text{and} \quad xdS + yR = Cd \]

\[ \Rightarrow x = \frac{(Cu - Cd)}{S(u - d)} \]
\[ \Rightarrow y = \frac{(1/R) [uCd - dCu]}{(u - d)} \]

\[ \text{P’folio Value = Option Price} = \frac{[(R - d)Cu + (u - R)Cd]}{R(u-d)} \]

For Example Problem
\[ = \frac{[(1.1 - 0.8) (15) + (1.25 - 1.1)(0)]}{1.1(1.25 - 0.8)} \]
\[ = \frac{[0.3(15)]}{1.1(0.45)} = \frac{10}{1.1} = 9.09 \quad \text{as before} \]

**Reformulation of Binomial Formulation**

- Rewrite formula, using factor “q” = \( \frac{(R - d)}{(u - d)} \)

- Option Price = \[ \frac{[(R - d)Cu + (u - R)Cd]}{R(u-d)} \]
  = \( \frac{(1/R) [qCu + (1-q) Cd]}{1}\)

- This leads to an extraordinary interpretation!
  Value of option = “expected value” with binomial probabilities \( q \) and \( 1 - q \) “risk-neutral probabilities”

- Yet “q” is defined in terms spread (as above) and, as we saw, actual probabilities do not enter into calculation!

\[
\begin{array}{c}
\text{Max}(\text{Su} - \text{K}, 0) = \text{Cu} \\
\text{Max}(\text{Sd} - \text{K}, 0) = \text{Cd}
\end{array}
\]
General Binomial Model Procedure

- Very much like a multi-period decision tree, using $q$ and $(1-q)$ on each branch

- Analysis is comparable
  - Work backward from date of expiration of option
  - For each period, apply one-period valuation

- At each node, compare
  - Value of option versus Immediate exercise payoff

- ... and Determine Optimal decision
  - Hold option for another period
  - Exercise immediately

- Main issue: values of “$u$” and “$d$”, that specify “$q$”

Determining “$u$” and “$d$”

- These parameters come from assumption of random distribution of variations around a trend

- Usual assumption is that random variations occur on a log-normal scale, with standard deviation $\sigma$

\[
u = e^{\exp(\sigma\sqrt{\Delta t})} \\
d = e^{\exp(-\sigma\sqrt{\Delta t})}
\]

$\Delta t$ is the fraction of a year (e.g, 1 month = 1/12), so that $R = 1 + Rf \times \Delta t$
“Lattice” for General Binomial Model

- “Lattice” is usual term for Binomial “decision tree”

<table>
<thead>
<tr>
<th>Period 0</th>
<th>Period 1</th>
<th>Period 2</th>
<th>Period 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>Cu</td>
<td>Cuu</td>
<td>Cuuu</td>
</tr>
<tr>
<td></td>
<td>Cd</td>
<td>Cud</td>
<td>Cuud</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Cdd</td>
<td>Cudd</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Cddd</td>
</tr>
</tbody>
</table>

- Work backward from last to first period to value C
- Apply 1 - period analysis at each node, for example:

\[ C_{uu} = \text{Max}\{S_{uu}-K, \left[q^* C_{uuu} + (1-q)^* C_{uud}\right]/(1+r), 0\} \]

Summary for Binomial Model

- Binomial model is a recursive technique
  - Start with end-period values and work backward to present
  - Tedious, but usually automated in computer programs
- Note similarity to NPV
  - Estimate cash-flows (end-of-period option value)
  - Discount to present (using risk-free rate)

\[ C = [q^* C_u + (1-q)^* C_d]/(1+r) \]

- But Model is very different from NPV analysis!
  - Payoffs are created by the factors “u” and “d”
  - The “probability” “q” does not represent actual probabilities;
    it is derived from Arbitrage-enforced pricing
  - The discount rate is “risk free” – because based on Arbitrage
Summary

- Value of options INCREASES WITH RISK

- Value of Options is determined by Arbitrage -- This is the fundamental observation

- Arbitrage creates portfolio to match option payoffs -- being risk-free, evaluated at risk-free rate

- Black-Scholes formula is compact, but limited
  - Values European calls, not generally applicable to design

- Binomial model more general
  - More complicated, but can be automated
  - Like decision trees
  - Uses "risk-neutral probabilities"