Option Valuation (Lattice)

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Outline

- Recall: Types of Options Mantra
- Lattice Method of Representing Uncertainty
  - Formulation and Key attributes
- Application to Decision Analysis
- Application to Financial Options
  - Portfolio Replication interpreted as “probabilities”
  - Arbitrage possibility enforces risk free rate
  - Analysis with $R_f$ and “risk neutral probabilities”
Two Types of Options

- **Financial Options**
  - These concern contracts on traded assets (such as stock, bonds, foreign exchange, bonds, etc.)
  - These are most common
  - Largest, oldest (to 1970s) and most sophisticated literature

- **“Real” Options**
  - These concern projects which may or may not produce a traded asset (Ex: a copper mine)
  - Least talked about
  - Recent literature (most from 1990’s)

Two Types of Real Options

- **Options “ON” projects**
  - These do not concern themselves with system design
  - They examine options to open, close, delay a project
  - EX: the option to open a mine (Antamina case)
  - Most common in literature

- **Options “IN” projects**
  - These involve changing the technology some way
  - These require detailed understanding of system
  - EX: design elements that permit changing altitude (and capacity) of communication satellites
  - Most interesting to system designers
Total: Three Types of Options

- Financial options
- Options ON projects
- Options IN projects

Real Options

- These distinctions effect the valuation method
- The question is: which approaches best in each circumstance?

Valuing Options: Concepts

- Financial Options:
  - ONLY Correct Way: “Arbitrage Enforced Pricing”
    * Ability to construct replicating portfolio
    * this portfolio defines price that must prevail
    * If not, others in market can make you a loser

- Real Options:
  - Multiple approaches (see Borison), for example:
    * Based on replicating portfolio if data good (Merck)
    * Decision analysis if no historical data (Kodak)
    * Simulation using either or both (Antamina)
  - Lattice analysis a key tool in overall context
Lattice Method (Binomial Tree)

- Reproduces uncertainty over time to simulate actual sequence of possibilities

- Approximates price changes as sequence of increases and decreases over stages

- Accuracy depends on number of stages – can be very detailed and accurate

- Has special features that permit easy solution using Dynamic Programming

Lattice Construction: 1 Stage module

- Lattice is a sequence of single stage modules

- Each shows changes in state of system

- Binomial if only 2 possibilities: up or down changes with probability, $P_u$ and $P_d$

- State of system correspondingly changes by a multiplicative factor up or down, $u$ or $d$
Lattice: Many stages (Decision Tree)

- Lattice is sequence of single-stage modules
  - States coincide ("up then down" path gives same state as "down then up")
  - Number of states increases linearly (1, 2, 3, 4, ...), not exponentially (1, 2, 4, 8, ...)
  - System state defines Node – PATH INDEPENDENT

Implications of Path Independency

- P.I. permits implicit enumeration of paths
  - You do not have to examine all paths to a node
  - E.g.: Best decision at Cud defined only by state of system, not by paths to Cud. When you have found it, you have implicitly considered paths to get there
  - Analysis linear in # of stages (not exponential)
Analysis of Lattice

- Lattice analysis is as for decision tree:
  - Start at end (right-hand side)
  - Determine best choice at that stage
  - Roll back to previous stage, and repeat

- When P.I. holds, a special efficient method is possible: Dynamic Programming (see text)
  - Defines Recursion formula that expresses one stage as function of next (Bellman Equation)
  - Automates process

- Can be done with Excel add-ins

Application to Decision Analysis

From Homework: Optimal Plant Investment (2)

<table>
<thead>
<tr>
<th>Demand Level</th>
<th>Plan B</th>
<th>Plan B+</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>1.40</td>
<td>-3.66</td>
</tr>
<tr>
<td>Medium</td>
<td>12.49</td>
<td>18.53</td>
</tr>
<tr>
<td>High</td>
<td>13.51</td>
<td>22.57</td>
</tr>
</tbody>
</table>

Why should analysis be limited to a decision in year 3? To only high, medium and low demand scenarios?
More detail with Decision Analysis

Can be done
- Make $\Delta t$ smaller
- Over $\Delta t$, project demand up or down
- Over entire horizon, demand can cover a known distribution

But
- Tree $\Rightarrow$ messy bush
- Even for simple case

Solution by Lattice Analysis

Analyze lattice, get results as shown below
- Nodes show state (demand over option value)
Application to Financial Options

- The correct analysis of financial options requires arbitrage enforced pricing
  - Previously used example demonstrates point

- Thus, traditional probabilities (defined by logic, frequency or estimates) NOT appropriate

- How then does lattice analysis work?

- Answer is TRICKY! Sets up portfolio to look like probabilities, but the are not really!

Case from previous lecture

- Idea: determine Fair Value of Option, $C$
  -- If end of period asset price $S > K$, strike price: payoff of the option = $S - K$
  -- If end of period asset price $S < K$, strike price: payoff of the option = 0

<table>
<thead>
<tr>
<th></th>
<th>Start</th>
<th>End</th>
<th>End</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset Price</td>
<td>100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Buy Call Strike = 110</td>
<td>- $C$</td>
<td>0</td>
<td>(125 - 110) = 15</td>
</tr>
</tbody>
</table>
### Table of Portfolio Cost and Payoffs

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<td>Asset price</td>
<td>100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Buy Stock</td>
<td>-100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Borrow Money</td>
<td>80/(1+r)</td>
<td>- 80</td>
<td>- 80</td>
</tr>
<tr>
<td>Net</td>
<td>-100 + 80/(1+r)</td>
<td>0</td>
<td>45</td>
</tr>
</tbody>
</table>

The Portfolio of Buying Asset with Loan replicates option

### Comparing Costs and Payoffs of Option and Replicating Portfolio

- If $S < K$, both payoffs = 0 and are equal
- If $S > K$, portfolio payoff is a multiple of call payoff. In this case, ratio is 3:1
- Thus, payoff of 3 calls = portfolio payoff
- Note: arbitrage is riskless so $r = \text{risk-free rate}$

<table>
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<th>Period</th>
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<th>End</th>
<th>End</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset Price</td>
<td>100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Buy Call</td>
<td>- C</td>
<td>0</td>
<td>(125-110) = 15</td>
</tr>
<tr>
<td>Buy Asset And Borrow</td>
<td>-100 + 80/(1 + r)</td>
<td>0</td>
<td>45</td>
</tr>
</tbody>
</table>
Value of Option

- Value of Option = Value of Portfolio
- This is easy to define, using risk-free rate, Rf
  - Calculation below assumes Rf = 10% (for easy calculation)
- \[ C = \left(\frac{1}{3}\right) \left[ -100 + \frac{80}{(1 + Rf)} \right] = $9.09 \]

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</tr>
</thead>
<tbody>
<tr>
<td>Asset Price</td>
<td>100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Buy 3 Calls</td>
<td>-3C</td>
<td>0</td>
<td>45</td>
</tr>
<tr>
<td>Buy Asset And Borrow</td>
<td>-100 + 80/(1 + r)</td>
<td>0</td>
<td>45</td>
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Interpreting Example as 1-stage lattice

- Value of Option is:
  - At Start: C this value is to be found
  - At End: either Cu = 15 or Cd = 0 (uncertain outcomes)
- Likewise, value of Asset is:
  - At Start: S
  - At End: either uS or dS
- To find C, we have to find share of asset ("x") and loan ("y") in replicating portfolio
- We solve: \[ xuS + yR = Cu \quad \text{and} \quad xdS + yR = Cd \]
  \[ \Rightarrow x = \frac{(Cu - Cd)}{S(u - d)} \]
  \[ \Rightarrow y = \frac{(1/R) [uCd - dCu]}{(u - d)} \]
Solving Lattice for Value of Option

Portfolio Value = Option Price
\[ \frac{[(R - d)Cu + (u - R)Cd]}{R(u-d)} \]
\[ = \frac{(1.1 - 0.8)(15) + (1.25 - 1.1)(0)}{1.1(1.25 - 0.8)} \]
\[ = \frac{0.3(15)}{1.1(0.45)} = \frac{10}{1.1} \]
\[ = 9.09 \text{ as before} \]

Rewrite formula, using factor “q” = \( \frac{R - d}{u - d} \)

Option Price = \[ \frac{(R - d)Cu + (u - R)Cd}{R(u-d)} \]
\[ = \frac{1}{R} \left[ qCu + (1-q)Cd \right] \]

Note that this looks just like a probabilistic binomial stage with probability “q” and “1-q” !!

Arbitrage seen as Expected Value

- Extraordinary interpretation!
- Option value = “expected value” over “risk-neutral probabilities” q and (1 - q)
- Yet “q” is defined in terms of spread
- Actual probabilities do not enter calculation!

- Graphically:

\[ C \]
\[ q \] \( \text{Max}(S_u - K, 0) = Cu \)
\[ (1-q) \] \( \text{Max}(S_d - K, 0) = Cd \)
General Multi-Stage Procedure

- Lattice analysis using $q$ and $(1 - q)$

- At each node, compare:
  - Option value
  - Payoff of Immediate exercise

- ... and Determine Optimal decision
  - Hold option for another period
  - Exercise immediately

- Issue: values of “$u$” and “$d$”, that specify “$q$”

Determining “$u$”, “$d$”: Assumptions

- These parameters reflect range of possible outcomes, thus an assumed pdf
- Usual assumption is that pdf is random, because
  - Project risks can be avoided by diversification
  - Thus only looks at market risk
  - Assumes efficient markets thus no bias
  - Thus error is random or “white noise”
- Accepts that
  - growth trends may exist
  - Values do no go negative
- Thus, usual assumption is that random variations is
  - log-normal, with standard deviation $\sigma$
  - Wiener process or Generalized Brownian Motion
Determining “u”, “d”: Formulas

- With Usual assumption that random variations are log-normal scale, with standard deviation $\sigma$
  
  $u = e^{\exp(\sigma \sqrt{\Delta t})}$
  
  $d = e^{\exp(-\sigma \sqrt{\Delta t})}$

Where $\Delta t$ is the fraction of a year (e.g., 1 month = 1/12), so that

$R = 1 + R_f \ast \Delta t$

Summary for Application to Financial Options

- Binomial model is a recursive technique
  - Starts with end-period values, works back to present
  - Tedious, but usually automated

- Note similarity to NPV
  - Estimate cash-flows (end-of-period option value)
  - Discount to present (using risk-free rate)

- But Model is very different from NPV analysis!
  - Payoffs are created by the factors “u” and “d”
  - The “probability” “q” is not an actual probability; it is derived from Arbitrage-enforced pricing
  - Discount rate is “risk free” – due to Arbitrage
Summary

- Lattice Method similar to a Decision Tree,
  - … but with specific structure
    - Nodes coincide
    - Values at nodes defined by State of System
    - Thus “path independent” values
    - Enabling rapid analysis (Dynamic Programming)

- Lattice Analysis widely applicable
  - With actual probability distributions
  - For Financial options using factors that are
    - called “risk-neutral probabilities”
    - But actually represent relative share of loan and stock in replicating portfolio