Marginal Analysis

Purposes:

1. To present a basic application of constrained optimization

2. Apply to Production Function to get criteria and patterns of optimal system design

Marginal Analysis: Outline

1. Definition and Assumptions
2. Optimality criteria
   - Analysis
   - Interpretation
   - Application
3. Key concepts
   - Expansion path
   - Cost function
   - Economies of scale
4. Summary
Marginal Analysis: Concept

- Basic form of optimization of design
- Combines:
  - Production function - Technical efficiency
  - Input cost function, c(X)
  - Economic efficiency

Marginal Analysis: Assumptions

- Feasible region is convex (over relevant portion) This is key. Why?
  - To guarantee no other optimum missed
- No constraints on resources
  - To define a general solution
- Models are “analytic” (continuously differentiable)
  - Finds optimum by looking at margins -- derivatives
Optimality Conditions for Design, by Marginal Analysis

The Problem:
Min \( C(Y') = c(X) \) cost of inputs for any level of output, \( Y' \)

s.t. \( g(X) = Y' \) production function

The Lagrangean:
\[ L = c(X) - \lambda [g(X) - Y'] \]

Optimality Conditions for Design:
Results

- Key Result:
\[ \frac{\partial c(X)}{\partial X_i} = \lambda \frac{\partial g(X)}{\partial X_i} \]

- Optimality Conditions:
\[ \frac{MP_i}{MC_i} = \frac{MP_j}{MC_j} = 1 / \lambda \]
\[ \frac{MC_j}{MP_j} = \frac{MC_i}{MP_i} = \lambda = \text{Shadow Price on Product} \]

- A balanced design
Each \( X_i \) contributes “same bang for buck”
Optimality Conditions: Graphical Interpretation of Costs

(A) Input Cost Function

Linear Case:
B = Budget
c(X) = Σp_iX_i ≤ B

General case:
Budget is non-linear (as in curved line)

(B) Conditions

Slope = MRS_{ij} = - MP_1 / MP_2
= - MC_1 / MC_2
Application of Optimality Conditions -- Conventional Case

Problem: \( Y = a_0 X_1^{a_1} X_2^{a_2} \)
\[ c(X) = \sum p_i X_i \]

Note: Linearity of Input Cost Function
- typically assumed by economists
- in general, not valid
  
  prices rise with demand
  wholesale, volume discounts

Solution:
\[ \left[ \frac{a_1}{X_1^*} \right] \frac{Y}{p_1} = \left[ \frac{a_2}{X_2^*} \right] \frac{Y}{p_2} \]
\[ \Rightarrow \quad \left[ \frac{a_1}{X_1^*} \right] \frac{p_1}{p_2} = \left[ \frac{a_2}{X_2^*} \right] \]

( * denotes an optimum value)

Optimality Conditions: Example

Assume \( Y = 2X_1^{0.48} X_2^{0.72} \) (increasing RTS)
\[ c(X) = 5X_1 + 3X_2 \]

Apply Conditions: \( = \frac{MP_i}{MC_i} \)
\[ \left[ \frac{a_1}{X_1^*} \right] / p_1 = \left[ \frac{a_2}{X_2^*} \right] / p_2 \]
\[ 0.48/ 5 / X_1^* = [ 0.72/ 3 ] / X_2^* \]
\[ 9.6 / X_1^* = 24 / X_2^* \]

This can be solved for a general relationship between resources \( \Rightarrow \) Expansion Path
Expansion Path

- Locus of all optimal designs \( X^* \)
- Not a property of technical system alone
- Depends on local prices
- Optimal designs do not, in general, maintain constant ratios between optimal \( X_1^* \)
  
  Compare: crew of 20,000 ton ship
  crew of 200,000 ton ship
  larger ship does not need 10 times as many sailors as smaller ship

Expansion Path: Non-Linear Prices

- Assume: \( Y = 2X_1^{0.48}X_2^{0.72} \) (increasing RTS)
  \( c(X) = X_1 + X_2^{1.5} \) (increasing costs)

- Optimality Conditions:
  \[
  \frac{0.48}{X_1} \frac{Y}{1} = \frac{0.72}{X_2} \frac{Y}{1.5X_2^{0.5}}
  \]
  \[\Rightarrow X_1^* = (X_2^*)^{1.5}\]

- Graphically:
Cost Functions: Concept

- Not same as input cost function
  It represents the optimal cost of \( Y \)
  Not the cost of any set of \( X \)

- \( C(Y) = C(X^*) = f(Y) \)

- It is obtained by inserting optimal values of resources (defined by expansion path) into input cost and production functions to give “best cost for any output”

Cost Functions: Graphical View

- Graphically:

- Great practical use:
  How much \( Y \) for budget?
  \( \Delta Y \) for \( \Delta B \)?
  Cost effectiveness, \( \Delta B / \Delta Y \)
Cost Function Calculation: Linear Costs

- Cobb-Douglas Prod. Fcn: \( Y = a_0 \pi X_i^{a_i} \)
- Linear input cost function: \( c(X) = \sum p_i X_i \)
- Result
  \[ C(Y) = A(\pi p_i^{a_i/r}) Y^{1/r} \quad \text{where} \quad r = \sum a_i \]

- Easy to estimate statistically
  => Solution for \( a_i \)
  => Estimate of prod. fcn. \( Y = a_0 \pi X_i^{a_i} \)

Cost Function Calculation: Example

- Assume Again: \( Y = 2X_1^{0.48} X_2^{0.72} \)
  \( c(X) = X_1 + X_2^{1.5} \)
- Expansion Path: \( X_1^* = (X_2^*)^{1.5} \)

- Thus:
  \( Y = 2(X_2^*)^{1.44} \)
  \( c(X^*) = 2(X_2^*)^{1.5} \)
  => \( X_2^* = (Y/2)^{0.7} \)
  \( c(Y) = c(X^*) = (2^{-0.05}) Y^{1.05} \)
Real Example: Communications Satellites

- System Design for Satellites at various altitudes and configurations
- Source: O. de Weck and MIT co-workers
- Data generated by a technical model that costs out wide variety of possible designs
- Establishes a Cost Function
- Note that we cannot in general expand along cost frontier. Technology limits what we can do: Only certain pathways are available

Key Parameters

- Each star in the Trade Space corresponds to a vector:
  - \( r \): altitude of the satellites
  - \( \varepsilon \): minimum acceptable elevation angle
  - \( C \): constellation type
  - \( P_t \): maximal transmitter power
  - \( D_A \): Antenna diameter
  - \( \Delta f_c \): bandwidth
  - \( \text{ISL} \): intersatellite links
  - \( T_{sat} \): Satellites lifetime
- Some are fixed:
  - \( C \): polar
  - \( \Delta f_c \): 40 kHz
Trade Space

Path example

Constants:
Pt = 200 W
DA = 1.5 m
ISL = Yes
T_{sat} = 5

System Lifetime capacity [min]
**Economies of Scale: Concept**

- A possible characteristic of cost function

- Concept similar to returns to scale, **except**
  - ratio of \(X_i\) not constant
  - refers to costs (economies) not “returns”
  - Not universal (as RTS) but depends on local costs

- **Economies of scale exist if costs increase slower than product**
  \[
  \text{Total cost} = C(Y) = Y^\epsilon \quad \epsilon < 1.0
  \]
Economies of Scale: Specific Cases

- If Cobb-Douglas, **linear input costs**, Increasing RTS \( r = \sum a_i > 1.0 \)

  \[ \Rightarrow \text{Economies of scale} \]

  Optimality Conditions: \( \frac{[a_1/X_1^*]}{p_1} = \frac{[a_2/X_2^*]}{p_2} \)

  Thus: Inputs Cost is a function of \( X_1^* \)

  Also: Output is a function of \( [X_1^*]^r \)

  So: \( X_1^* \) is a function of \( Y^{1/r} \)

  Finally: \( C(Y) = \text{function of } [Y^{1/r}] \)

Economies of Scale: General Case

Not necessarily true in general that Returns to scale \( \Rightarrow \) Economies of Scale

Increasing costs may counteract advantages of returns to scale

See example!!

\[ c(Y) = c(X^*) = (2^{-0.05})Y^{1.05} \]

Contrarily, if unit prices decrease with volume (quite common) we can have Economies of Scale, without Returns to Scale
Marginal Analysis: Summary

- Assumptions --
  -- convex feasible region
  -- Unconstrained

- Optimality Criteria
  - MC/MP same for all inputs

- Expansion path -- Locus of Optimal Design

- Cost function -- cost along Expansion Path

- Economies of scale (vs Returns to Scale)
  -- Exist if Cost/Unit decreases with volume