Linear Programming
Sensitivity Analysis

- Rationale
- Shadow Prices
  - Definition
  - Use
  - Sign
  - Range of Validity
- Opportunity Costs
  - Definition
  - Use
Rationale for Sensitivity Analysis

- **Math problem is an approximation**
  - optimum is an approximation
  - we need to check

- **Constraints often artificial**
  - Designer should question them
  - *Should we have different specifications?*

- **Situations always probabilistic**
  - Prices change
  - Need to assess risk

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Shadow Price Definition

- **Recall from Constrained Optimization:**
  - Shadow price = $\frac{\delta \text{ (objective function)}}{\delta \text{ (constraint)}}$ at the optimum

  - Complementary Slackness:
    - Either (Slack variable) or (shadow price) = 0
Shadow Price Illustration

Max: \[ Y = X_1 + 4X_2 \]
\[ \text{s.t.} \]
\[ X_1 + X_2 \leq 5 = b_1 \]
\[ X_1 \geq 3 = b_2 \]
\[ X_2 \leq 3 = b_3 \]
\[ X_1, X_2 \geq 0 \]

Notes:
\[ a) \] \[ X_1^* = 3; X_2^* = 2; Y^* = 11 \]
\[ b) \] when \[ \Delta b_1 = \pm 1 \]
\[ \Delta X_1^* = 0; \Delta X_2^* = \pm 1 \]
\[ \Delta Y^* = \pm 4; \ SP_1 = 4 \]
\[ c) \] \[ SP_3^* = 0; \text{slack}_3 = 1 \]
d\[ d) \] when \[ b_1 > 6 \]
\[ \text{slack}_3 = 0; \ SP_3 \neq 0; \]
\[ SP_1 = 1 \leq 4 \]

Proactive Use of Shadow Prices

- Identify constraints with high S.P
- See if they can be changed for better solutions
- Example: New York water supply
  - Original Design for Third City Tunnel ($1 billion plus)
  - pressure < 40 psi at curb (some point in Brooklyn)
  - No allowance for local tanks, pumps
  - Shadow price in millions of dollars!
Reactive Use of Shadow Prices

- Respond to new opportunities
- Example: client changes specifications

- Respond to proposals for new constraints
- Example: trace chemicals

Sign of Shadow Prices

- "Obvious Rule" (+SP with +Δb) not correct

- Correct Reasoning:
  - What makes the optimum better?
    • Expansion of feasible region => "Relaxation of constraints"
  - What changes will increase the feasible region?
    • Increase upper bound \( \sum_j a_{ij}x_j < b_i \)
    • Decrease lower bound \( \sum_k a_{kj}x_j > b_k \)
  - i.e., "Raise the roof, lower the floor."
Shadow Price Illustration

Max: \( Y = X_1 + 4X_2 \)
s.t. \( X_1 + X_2 \leq 5 = b_1 \)
\( X_1 \geq 3 = b_2 \)
\( X_2 \leq 3 = b_3 \)
\( X_1, X_2 \geq 0 \)

Notes:

a) \( X_1^* = 3; \ X_2^* = 2; \ Y^* = 11 \)
b) when \( \Delta b_1 = \pm 1 \)
\( \Delta X_1^* = 0 ; \ \Delta X_2^* = \pm 1 \)
\( \Delta Y^* = \pm 4 ; \ SP_1 = 4 \)
c) \( SP_3^* = 0; \ slack_3 = 1 \)
d) when \( b_1 > 6 \)
slack_3 = 0; \ SP_3 \neq 0;
\( SP_1 = 1 \leq 4 \)

Shadow Prices As Constraints Change

increase an upper bound ("raise the roof")
decrease a lower bound ("lower the floor")

Engineering Systems Analysis for Design
Richard de Neufville, Joel Clark and Frank R. Field
Massachusetts Institute of Technology
LP Sensitivity Analysis Slide 10 of 22
Range of Shadow Prices

- In Linear Programming, Shadow prices are constant
- Until a constraint changes enough so that a new constraint is binding
- Results given as:
  \[ SP_K = \text{constant} \]
  \[ \text{for } r_L < b_K < r_U \]
- Outside the range:
  - Shadow prices decrease as constraint is relaxed
  - Shadow prices increase as constraint is tightened

Shadow Price Ranges for Example

Max: \[ Y = X_1 + 4X_2 \]
\[ \text{s.t. } X_1 + X_2 \leq 5 = b_1 \]
\[ X_1 \geq 3 = b_2 \]
\[ X_2 \leq 3 = b_3 \]
\[ X_1, X_2 \geq 0 \]

Shadow Prices
\[ SP_1 = 4 \]
\[ 3 \leq b_1 \leq 6 \]
\[ SP_2 = 3 \]
\[ 2 \leq b_2 \leq 5 \]
\[ SP_3 = 0 \]
\[ 2 \leq b_3 \]
Opportunity Cost - Definition

- Objective Function = \( \sum c_i X_i \)
- Opportunity costs associated with \( c_i \) -- the coefficients of design/decision variables
- At optimum, some decision variables = 0
  - These are non-optimal decision variables
- Opportunity cost is:
  - Degradation of optimum per unit of non-optimal variable introduced into design
  - A "cost" in that it is a worsening of optimum. Units may be almost anything; equal to whatever units are being optimized.

Meaning of Opportunity Costs

- Opportunity cost defines design trigger "price"
  - The value of the coefficient of the decision variable for which that variable should be in the design
- Suppose: Obj.Function = ... + \( c_K X_K + ... \)
  and \( X_K \) not optimal with an opportunity cost = \( OC_K \)
- Then, as \( c_K \) changes for the better, (greater for maximization, lesser for minimization)
  - \( OC_K \) lower
  - \( OC_K = 0 \) at \( c'_K = c_K - OC_K \)
- \( c'_K \) is trigger price; defines the limit of best design
Illustration of Opportunity Cost

- What happens when forced to use a non-optimal decision variable?
- Example: Min Cost = 2X₁ + 10X₂ + 20X₃
  \[
  \begin{align*}
  &\text{s.t. } X₁ + X₂ + X₃ \geq 3 \\
  &X₂ \geq 1 \\
  &X₁, X₂, X₃ \geq 0
  \end{align*}
  \]
  - X* = (2, 1, 0); cost* = 14
  - If forced to use X₃, new X* = (1,1,1); new cost* = 32
  Thus: (opportunity cost)_3 = ΔZ*/1 = 18

Use of Opportunity Cost

- At what price would it be desirable to use X₃?

- If X₃ is used with no change in its unit cost (c₃), the optimal cost would increase by 18
- If the cost of X₃ were to fall by an amount equal to the opportunity cost (c₃' = c₃ - OC₃ = 20 - 18 = 2). It would then compete with X₁
- So the answer is: When its unit cost falls by its opportunity cost: 20 - 18 = 2
How do you find SP and OC?

- LP optimization programs all calculate shadow prices and opportunity costs routinely and “print them out” for you.
- Sometimes, programs report this information in special ways. Thus:
  - Shadow Prices => Shadow prices or “dual decision variables”
  - Opportunity Costs => Reduced Gradient or “dual slack variables”
  - More on “duality” later.

A Possible Semantic Confusion

- Note that the Phrases “shadow price” and “opportunity cost” have somewhat different meanings in LP and Economics literature.
- The “opportunity cost” of an action in economics can be interpreted as the “shadow price” of that action on the budget...
Example for Assumptions

minimize: \( Z = 3X_1 + 5X_2 \)

s.t.  
\[ X_1 + X_2 \geq 5 \]
\[ 3X_1 + 2X_2 \leq 18 \]
\[ 6X_1 + 5X_2 \leq 42 \]
\[ -7X_1 + 8X_2 \leq 0 \]
\[ 0 \leq X_1 \leq 4 \]
\[ 0 \leq X_2 \leq 6 \]

Solution:
\[ X_1^* = 4 \quad X_2^* = 1 \]

Example Solution from Excel Solver

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<th>CONSTRAINT</th>
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Example Sensitivity Report from Excel Solver

Adjustable Cells

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Constraints

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Summary on LP Sensitivity Analysis

- LP Optimization Programs automatically provide important information useful for improving/changing design
- Shadow prices -- to help redefine constraints
- Opportunity costs -- to identify critical prices
- Need to understand these quantities carefully