Constrained Optimization

- Objective of Presentation: To present the basic conceptual methods used to optimize design in real situations

- Essential Reality: In practical situations, the designers are constrained or limited by
  - physical realities
  - design standards
  - laws and regulations, etc.

Outline

- Unconstrained Optimization (Review)
- Constrained Optimization
  - Approach
  - Equality constraints
    - Lagrangeans
    - Shadow prices
  - Inequality constraints
    - Kuhn-Tucker conditions
    - Complementary slackness
Unconstrained Optimization: Definitions

- Optimization => Maximum of desired quantity, or => Minimum of undesired quantity
- Objective Function = Expression to be optimized = Z(\(X\))
- Decision Variables = Variables about which we can make decisions = \(X = (X_1,...,X_n)\)

Unconstrained Optimization: Graph

- B and D are maxima
- A, C and E are minima
Unconstrained Optimization: Conditions

- By calculus: if $F(X)$ continuous, analytic

- Primary conditions for maxima and minima:
  \[ \frac{\partial F(X)}{\partial X_i} = 0 \quad \forall i \]
  ( symbol means: for all $i$)

- Secondary conditions:
  \[ \frac{\partial^2 F(X)}{\partial X_i^2} < 0 \Rightarrow \text{Max} \quad (B,D) \]
  \[ \frac{\partial^2 F(X)}{\partial X_i^2} > 0 \Rightarrow \text{Min} \quad (A,C,E) \]
  These define whether point of no change in $Z$ is a maximum or a minimum

Unconstrained Optimization: Example

- Example: Housing insulation
  Total Cost = Fuel cost + Insulation cost
  $x = \text{Thickness of insulation}$
  \[ F(x) = \frac{K_1}{x} + K_2x \]

  Primary condition: \[ \frac{\partial F(x)}{\partial x} = 0 = -\frac{K_1}{x^2} + K_2 \]

  => $x^* = \left\{ \frac{K_1}{K_2} \right\}^{1/2}$
  (starred quantities are optimal)
Unconstrained Optimization: Graph of Solution to Example

- If: $K_1 = 500$ ; $K_2 = 24$ Then: $X^* = 4.56$

Constrained Optimization: General

- “Constrained Optimization” involves the optimization of a process subject to constraints

- Constraints have two basic types
  - Equality Constraints -- some factors have to equal constraints
  - Inequality Constraints -- some factors have to be less less or greater than the constraints (these are “upper” and “lower” bounds)
Constrained Optimization: General Approach

- To solve situations of increasing complexity, (for example, those with equality, inequality constraints) ...
- Transform more difficult situation into one we know how to deal with
- Note: this process introduces new variables!

- Thus, transform
  > “constrained” optimization to “unconstrained” optimization

Equality Constraints: Example

- Example: Best use of budget
- Maximize: Output = \( Z(X) = a_0 x_1^{a_1} x_2^{a_2} \)
- Subject to (s.t.):
  \[
  \text{Total costs} = \text{Budget} = p_1 x_1 + p_2 x_2
  \]

Note: \( \frac{\partial Z(X)}{\partial X} \neq 0 \) at optimum
Lagrangean Method: Approach

- Transforms equality constraints into unconstrained problem
- Start with:
  \[ \text{Opt: } F(x) \]
  \[ \text{s.t.: } g_j(x) = b_j \quad \Rightarrow \quad g_j(x) - b_j = 0 \]
- Get to:
  \[ L = F(x) - \sum_j \lambda_j [g_j(x) - b_j] \]
  \[ \lambda_j = \text{Lagrangean multipliers (lambdas) -- these are unknown quantities for which we must solve} \]

Note: \( [g_j(x) - b_j] = 0 \) by definition, thus optimum for \( F(x) \) = optimum for \( L \)

Lagrangean: Optimality Conditions

- Since the new formulation is a single equation, we can use formulas for unconstrained optimization.
- We set partial derivatives equal to zero for all unknowns, the \( X \) and the \( \lambda \)

- Thus, to optimize \( L \):
  \[ \frac{\partial L}{\partial x_i} = 0 \quad \forall_i \]
  \[ \frac{\partial L}{\partial \lambda_j} = 0 \quad \forall_j \]
Lagrangean: Example Formulation

- Problem:
  Opt: \( F(x) = 6x_1x_2 \)
  s.t.: \( g(x) = 3x_1 + 4x_2 = 18 \)

- Lagrangean:
  \[ L = 6x_1x_2 - \lambda(3x_1 + 4x_2 - 18) \]

Optimality Conditions:
\[
\begin{align*}
\frac{\partial L}{\partial x_1} &= 6x_2 - 3\lambda = 0 \\
\frac{\partial L}{\partial x_2} &= 6x_1 - 4\lambda = 0 \\
\frac{\partial L}{\partial \lambda} &= 3x_1 + 4x_2 - 18 = 0
\end{align*}
\]

Lagrangean: Graph for Example

Isoquants for \( F(X) = 20 \) and \( 40 \) and Constraint
Lagrangian: Example Solution

- Solving as unconstrained problem:
  \[ \frac{\partial L}{\partial x_1} = 6x_2 - 3\lambda = 0 \]
  \[ \frac{\partial L}{\partial x_2} = 6x_1 - 4\lambda = 0 \]
  \[ \frac{\partial L}{\partial \lambda} = 3x_1 + 4x_2 - 18 = 0 \]
  
- so that: \[ \lambda = 2x_2 = 1.5x_1 \] (first 2 equations)
  \[ \Rightarrow \ x_2 = 0.75x_1 \]
  \[ \Rightarrow \ 3x_1 + 3x_1 - 18 = 0 \] (3rd equation)

- \[ x_1^* = 18/6 = 3 \]
- \[ x_2^* = 18/8 = 2.25 \]
- \[ \lambda^* = 4.5 \]
- \[ F(x)^* = 40.5 \]
Shadow Prices

- Shadow Price is the Rate of change of objective function per unit change of constraint
  \[ \frac{\partial F(x)}{\partial b_j} \]

- It is extremely important for system design
- It defines value of changing constraints, and indicates if worthwhile to change them
  - Should we buy more resources?
  - Should we change environmental constraints?

Lagrangean Multiplier is a Shadow Price

- The Lagrangean multiplier is interpreted as the shadow price on constraint

  \[ SP_j = \frac{\partial F(x)}{\partial b_j} = \frac{\partial L}{\partial b_j} \]

  \[ = \frac{\partial \{ F(x) - \sum_j \lambda_j [g_j(x) - b_j] \} }{\partial b_j} = \lambda_j \]

  - Naturally, this is an instantaneous rate
Lagrangean = Shadow Prices
Example

- Let's see how this works in example, by changing constraint by 0.1 units:
  Opt: \[ F(x) = 6x_1 x_2 \]
  s.t.: \[ g(x) = 3x_1 + 4x_2 = 18.1 \]

- The optimum values of the variables are:
  \[ x_1^* = \frac{18.1}{6} \quad x_2^* = \frac{18.1}{8} \quad \lambda^* = 4.5 \]

- Thus \[ F(x)^* = 6\left(\frac{18.1}{6}\right)\left(\frac{18.1}{8}\right) = 40.95 \]

\[ \Delta F(x) = 40.95 - 40.5 = 0.45 = \lambda^* (0.1) \]

Inequality Constraints: Example

- Example: Housing insulation
  Min: Costs = \( K_1 / x + K_2 x \)
  s.t.: \( x \geq 8 \) (minimum thickness)

\[ \text{Optimizing Cost Example} \]

---

Inequality Constraints: Example

- Example: Housing insulation
  Min: Costs = \( K_1 / x + K_2 x \)
  s.t.: \( x \geq 8 \) (minimum thickness)
Inequality Constraints: Approach

- Transform inequalities into equalities, then proceed as before

- Again, this introduces new variables – in this case, the “Slack” variables that define “slack” or distance between constraint and amount used

- The resulting equations are known as the “Kuhn-Tucker conditions”

Inequality Constraints -- putting slack variables in Lagrangean

- A “slack variable”, $s_j$, for each inequality:
  \[ g_j(x) \leq b_j \Rightarrow g_j(x) + s_j^2 = b_j \]
  \[ g_j(x) \geq b_j \Rightarrow g_j(x) - s_j^2 = b_j \]

- These are “squared” to be positive

- Start from:
  \[ \text{opt: } F(x) \quad \text{s.t.: } g_j(x) \leq b_j \]

- Get to:
  \[ L = F(x) - \sum \lambda_j [g_j(x) + s_j^2 - b_j] \]
Inequality Constraints: Complementary Slackness Conditions

- The optimality conditions are:
  \[ \frac{\partial L}{\partial x_i} = 0 \]
  \[ \frac{\partial L}{\partial \lambda_j} = 0 \]
  plus: \[ \frac{\partial L}{\partial s_j} = 2s_j\lambda_j = 0 \]

- These new equations imply:
  \[ s_j = 0 \quad \lambda_j \neq 0 \]
  Or:
  \[ s_j \neq 0 \quad \lambda_j = 0 \]

These are "complementary slackness" conditions.

Either slack or lambda (or both) = 0 \quad \forall \, j

Interpretation of Complementary Slackness Conditions

- Interpretation:

  - If there is slack on \( b_j \), \( \lambda_j = 0 \) (i.e. more than enough of it)
    \[ \Rightarrow \text{No value to objective function} \]
    to having more: \( \lambda_j = \frac{\partial F(x)}{\partial b_j} = 0 \)

  - If \( \lambda_j \neq 0 \), then all available \( b_j \) used
    \[ \Rightarrow s_j = 0 \]
    In this case, it would be worthwhile to
    have more of this constraint available
Example: Solution

- Min: Costs = \( K_1 / x + K_2 x \)
  s.t.: \( x \geq 8 \) (minimum thickness)
- Lagrangean: \( L = K_1/x + K_2 x - \lambda [x - s^2 - b] \)
  \[ = \frac{500}{x} + 24x - \lambda [x - s^2 - 8] \]
- Optimality Conditions:
  \[ \frac{\partial L}{\partial x_i} = \frac{500}{x^2} + 24 - \lambda = 0 \]
  \[ \frac{\partial L}{\partial s_j} = 2 \lambda s = 0 \]
  \[ \frac{\partial L}{\partial \lambda_j} = 0 \implies x - s^2 = 8 \]
- If \( s = 0, \ x = 8, \ \lambda = 31.8 \) (at that point)
  Min cost = 254.5

Example: Interpretation of Slack

- Since \( \lambda = 31.8 > 0 \)
- Therefore, worth changing constraint to get better solution
  - In this case, the constraint is on minimum, so easing or relaxing it means that we would lower the constraint, setting a lower requirement on thickness of insulation
- Unconstrained solution for lifetime cost is:
  \( x^* = 4.56 \)
  Optimum = 221 < 254.5
Slack Variable Example: Graph

- The minimum cost with 8 inch constraint on thickness = 254.5
- If the constraint were less, optimum could be better (until unconstrained minimum)

![Optimizing Cost Example](image)

Another application to Example

- Min: Costs = $K_1 / x + K_2x$
  
s.t.: $x \geq 4$ (NEW MINIMUM)
- Lagrangean = $K_1/x + K_2x - \lambda [x + s^2 - b] = 500/x + 24x - \lambda [x + s^2 - 4]$
- Optimality Conditions:
  
  \[ \frac{500}{x^2} + 24 - \lambda = 0 \quad 2 \lambda s = 0 \quad x - s^2 = 4 \]
- If $\lambda = 0$, $x = 4.56$, slack, $s^2 = 1.56$
  
  Optimum = 221

- As $\lambda = 0$, not worth changing constraint
Summary of Presentation

- Important mathematical approaches
  - Lagrangeans
  - Kuhn- Tucker Conditions

- Important Concept: Shadow Prices

- THESE ANALYSES GUIDE DESIGNERS TO CHALLENGE CONSTRAINTS