Dynamic Programming and Replicating Portfolio Fundamentals

ESD.71 Fall 2004
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**Dynamic Programming**

- $g_i(X) -$ return function when state $X$ is applied to stage $i$
- Stage – time (periods), space (locations, travel), sequence (logistics), portfolio of investments, redundant components in a system
- State $X$ – set of possible choices for each stage (used in individual return function)
- $K$ – one of the possible states (used in cumulative return function)
- $f_s(K)$ - cumulative return function: what is best way to “be in” or “arrive at” State ($K$) over first ($S$) Stages?

- Both stages and states are sets.
- DP is applicable when the return function is discontinuous (step-wise).
- The recurrence formula gives the cumulative return function; it does not consider constraints.

Questions to ask yourself when approaching a problem:
- How can I best do __x__ in stage 1?
- How can I best do __x__ in stage 1 & 2?
- How can I best do __x__ in stage 1, 2, … & i?

**Ex 1. ASA text book Pr. 7.13**
Consider a truck whose maximum loading capacity is 10 tons. Suppose that there are 3 different items, various quantities of which are to make up a load to be carried to a remote geographical area. Suppose we wish to maximize the value of what the truck carries to the inhabitants, given the following weights and values of the four items.

<table>
<thead>
<tr>
<th>Item</th>
<th>Weight (tons)</th>
<th>Value ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3 t.</td>
<td>$5</td>
</tr>
<tr>
<td>2</td>
<td>4 t.</td>
<td>$7</td>
</tr>
<tr>
<td>3</td>
<td>5 t.</td>
<td>$8</td>
</tr>
</tbody>
</table>

- a) Find the optimal solution by dynamic programming.
- b) Write the recurrence formula for this problem
- c) Define the assumptions of dynamic programming.
Solution
Let the ‘states’ be the weight of the load. Construct a table of how to best load the truck in each stage, taken separately.

Table 1. How to best load the truck for each item taken separately.

<table>
<thead>
<tr>
<th>States X</th>
<th>g_1(X)</th>
<th>g_2(X)</th>
<th>g_3(X)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 t.</td>
<td>$0</td>
<td>$0</td>
<td>$0</td>
</tr>
<tr>
<td>1 t.</td>
<td>$0</td>
<td>$0</td>
<td>$0</td>
</tr>
<tr>
<td>2 t.</td>
<td>$0</td>
<td>$0</td>
<td>$0</td>
</tr>
<tr>
<td>3 t.</td>
<td>$5</td>
<td>$0</td>
<td>$0</td>
</tr>
<tr>
<td>4 t.</td>
<td>$5</td>
<td>$7</td>
<td>$0</td>
</tr>
<tr>
<td>5 t.</td>
<td>$5</td>
<td>$7</td>
<td>$8</td>
</tr>
<tr>
<td>6 t.</td>
<td>$10</td>
<td>$7</td>
<td>$8</td>
</tr>
<tr>
<td>7 t.</td>
<td>$10</td>
<td>$7</td>
<td>$8</td>
</tr>
<tr>
<td>8 t.</td>
<td>$10</td>
<td>$14</td>
<td>$8</td>
</tr>
<tr>
<td>9 t.</td>
<td>$15</td>
<td>$14</td>
<td>$8</td>
</tr>
<tr>
<td>10 t.</td>
<td>$15</td>
<td>$14</td>
<td>$16</td>
</tr>
</tbody>
</table>

Now calculate the following:
Step 1: \( f_1(K) \)  How can I best allocate \( K = 0, 1, 2, \ldots 10 \) tons in Stage 1?
For stage 1, the cumulative return function is simply the return function since we have only passed through one stage. See column ‘\( f_1(K) \)’ of Table 2.
Ex. \( f_1(3) = $5 \) (1 of item 1)

Table 2. Cumulative return functions for how to best load the truck.

<table>
<thead>
<tr>
<th>States K</th>
<th>( f_1(K) )</th>
<th>( f_2(K) )</th>
<th>( f_3(K) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 t.</td>
<td>$0</td>
<td>$0</td>
<td>$0</td>
</tr>
<tr>
<td>1 t.</td>
<td>$0</td>
<td>$0</td>
<td>$0</td>
</tr>
<tr>
<td>2 t.</td>
<td>$0</td>
<td>$0</td>
<td>$0</td>
</tr>
<tr>
<td>3 t.</td>
<td>$5</td>
<td>$5</td>
<td>$5</td>
</tr>
<tr>
<td>4 t.</td>
<td>$5</td>
<td>$7</td>
<td>$7</td>
</tr>
<tr>
<td>5 t.</td>
<td>$5</td>
<td>$7</td>
<td>$8</td>
</tr>
<tr>
<td>6 t.</td>
<td>$10</td>
<td>$10</td>
<td>$10</td>
</tr>
<tr>
<td>7 t.</td>
<td>$10</td>
<td>$12</td>
<td>$12</td>
</tr>
<tr>
<td>8 t.</td>
<td>$10</td>
<td>$14</td>
<td>$14</td>
</tr>
<tr>
<td>9 t.</td>
<td>$15</td>
<td>$15</td>
<td>$15</td>
</tr>
<tr>
<td>10 t.</td>
<td>$15</td>
<td>$17</td>
<td>$17</td>
</tr>
</tbody>
</table>
Step 2: \( f_2(K) \)  How can I best allocate \( K = 0, 1, 2, \ldots 10 \) tons in Stage 2 having passed through Stage 1?

Look at cumulative return function for stage 1 in addition to return function for stage 2.

\[
f_2(K) = \max [ g_2(X) + f_1(K-X) ]
\]

Ex. \( f_2(5) = \max [ (7 + 0), (0 + 5) ] = 7 \) (1 of item 2)

Step 3: \( f_3(K) \)  How can I best allocate \( K = 0, 1, 2, \ldots 10 \) tons in Stage 3 having passed through Stages 1 & 2?

Look at cumulative return function for stage 2 in addition to return function for stage 3.

\[
f_3(K) = \max [ g_3(X) + f_2(K-X) ]
\]

Ex. \( f_3(9) = \max [ ($0+$15), ($0 +$14), ($0+$12), ($5 +$10), ($7+$7), ($8+$7), ($10+$5), ($12 +$0), ($14 +$0), ($15 +$0) ] = $15 \)

(a) The composition of the truckload with maximum value when the maximum total weight is 10 t is to have (2) of item 1 and (1) of item 2, for a total value of $17 and total weight of 10 t. The next best load is worth $16, which consists of (2) of item 3 and a total weight of 10 t.

(b) The recurrence formula for this problem is

\[
f_i(K) = \max [ g_i(X) + f_{i-1}(K-X) ]
\]

where \( g_i(X) = XV_i \) and \( X \) is an integer 0, 1, 2, ..., \( n \); and \( V_i \) is the value of item \( i \).

The problem is constrained by a maximum weight of 10 tons, which is calculated as: \( XW_i \), where \( W_i \) is the weight of item \( i \).

(c) The assumptions for dynamic programming are

a. monotonicity, such that improvements in each return function lead to improvements in the objective function, and

b. separability, so that each return function is independent.

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**Replicating portfolio basics**

- Construct a table of payoffs of the option.
- Use the value of the asset in the down state to determine the size of the loan payoff.
- Alternative method to calculate call option value: use risk-neutral 'probability' (i.e. weights):

\[
q = (R – d)/(u-d)
\]

\[
C = (1/R)[q*C_u + (1-q)*C_d]
\]

\[
Where \ R = 1+r
\]

Ex. 2 (based on McDonald, 2003, pp. 309-311, 337-338):

For a current stock price of $41, one year possible future prices of $59.95 (up) and $32.90 (down), a call option with strike price $40 of expiring one year from now, and a risk-free rate of return of 8%,
a) construct the replicating portfolio for this call option and determine the value
b) determine the value of the call option using the risk-neutral ‘probability’ method
c) construct the arbitrage if the market gives a price of $8.50 for this option.

**Solution**

\[ S_0 = $41 \]
\[ S_u = $59.95 \]
\[ S_d = $32.90 \]
\[ T = 1 \text{ yr.} \]
\[ K = $40 \]
\[ r_f = 8\% \]

Calculate:
\[ u = \frac{S_u}{S_0} = \frac{59.95}{41} = 1.462 \]
\[ d = \frac{S_d}{S_0} = \frac{32.90}{41} = 0.802 \]

\[ C_u = \max [S_u - K, 0] = \max [(59.95 - 40), 0] = 19.95 \]
\[ C_d = \max [S_d - K, 0] = \max [(32.90 - 40), 0] = 0 \]

Table 3. Portfolio Costs and Payoffs

<table>
<thead>
<tr>
<th>Asset</th>
<th>Start (t=0)</th>
<th>End – up (t=1)</th>
<th>End – down (t=1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy S</td>
<td>41</td>
<td>59.95</td>
<td>32.90</td>
</tr>
<tr>
<td>Borrow (loan)</td>
<td>-30.46</td>
<td>59.95</td>
<td>32.90</td>
</tr>
<tr>
<td>Net</td>
<td>-10.54</td>
<td>27.05</td>
<td>0</td>
</tr>
</tbody>
</table>

Ratio of Value of option to value of portfolio at time (1) = 19.95/27.05 = 0.74
The value of the option (C) is this ratio multiplied by the time (0) value of the portfolio:
\[ C = 0.74(10.54) = \boxed{7.77} \]

Table 4. Value of Option Using Replicating Portfolio Method

<table>
<thead>
<tr>
<th>Asset Price</th>
<th>Start (t=0)</th>
<th>End – up (t=1)</th>
<th>End – down (t=1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 1 Call</td>
<td>-C</td>
<td>19.95</td>
<td>0</td>
</tr>
<tr>
<td>0.74 of portfolio of purchased asset and borrowed loan</td>
<td>-7.77</td>
<td>-19.95</td>
<td>0</td>
</tr>
<tr>
<td>Net</td>
<td>C=7.77</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
b) Alternative method to calculate call option value: use risk-neutral ‘probability’ (i.e. weights):

\[
q = \frac{1.08 - 0.802}{1.462 - 0.802} = 0.421 \\
1 - q = 0.579
\]

\[
C = \frac{1}{1.08}[(0.421)(19.95) + (0.579)(0)] = 7.78 = C
\]
(Result is slightly different due to rounding errors.)

c.) If the market price for the option is $8.50, instead of the theoretical price of $7.77, then we can make an arbitrage profit. That is, we can obtain money without having to invest any of our own. To make the profit, we could sell the option (thereby receiving $8.50 for something that is only worth $7.77), but that leaves us with the risk that the stock price will be in the ‘up’ state at expiration and we will have to deliver the stock. However, we can address the risk by buying a synthetic option at the same time we sell the actual option. The synthetic option consists of the portfolio we calculated earlier: buy 0.74 shares of the stock and borrow $30.46(0.74) = $22.54 (to partially fund the purchase of the stock).

If we simultaneously
   a) sell the actual option for $8.50, and
   b) buy the portfolio (synthetic option),
the initial (time 0, right now) cash flow is

\[
$8.50 - 0.74 \times $41 + $22.54 = $0.70
\]

We earn $0.70 for nothing! Now do that 1 million times… Note that we assume that we can frictionlessly sell the option, buy the stock, and obtain a loan.

Verify that there is no risk at expiration by hedging the option with the synthetic option (or portfolio).

\[
\begin{array}{|c|c|c|}
\hline
\text{Asset Price} & \text{End – up (t=1)} & \text{End – down (t=1)} \\
\hline
\text{Written call (pay if stock price goes up)} & -19.95 & 0 \\
\hline
\text{0.74 purchased shares} & 44.36 & 24.35 \\
\hline
\text{Repay loan with accumulated interest} & -24.35 & -24.35 \\
\hline
\text{Net Total Payoff} & 0 & 0 \\
\hline
\end{array}
\]

Reference: