Production Functions (PF)

Outline

1. Motivation
2. Definition
3. Technical Efficiency
4. Mathematical Representation
5. Characteristics

Production Function - Motivation

- In order to analyze a system, we need to model it, that is, provide connection between what we do, and what results
- Moreover, we need to focus our attention on the most interesting possibilities…
- This is role of “Production Function”
- It is basic Conceptual for Modeling Engineering Systems
Production Function - Definition

- Definition:
  - Represents technically efficient transform of physical resources $X = (X_1, ..., X_n)$ into product or outputs $Y$ (may be good or bad)

- Example:
  - Use of aircraft, pilots, fuel (the $X$ factors) to carry cargo, passengers and create pollution (the $Y$)

- Typical focus on 1-dimensional output

Technical Efficiency

- A Process is Technically Efficient if it provides Maximum product from a given set of resources $X = X_1, ..., X_n$

- Graph:
  ![Graph](image-url)
Mathematical Representation -- General

- Two Possibilities
  - Deductive -- Economic
    - Standard economic analysis
    - Fit data to convenient equation
    - Advantage - ease of use
    - Disadvantage - poor accuracy
  - Inductive -- Engineering
    - Create system model from knowledge of details
    - Advantage - accuracy
    - Disadvantage - careful technical analysis needed

Mathematical Representation -- Deductive

- Standard Cobb-Douglas Production Function $Y$
  
  $Y = a_0 \pi x_1^{a_1} \ldots x_n^{a_n}$  \[ \pi \text{ means multiplication} \]
  
  - Interpretation: ‘$a_i$’ are physically significant
  - Easy estimation by linear least squares
    
    $\log Y = \log a_0 + \sum a_i \log X_i$

- Translog PF -- more recent, less common

  $\log Y = a_0 + \sum a_i \log X_i + \sum a_{ij} \log X_i \log X_j$
  
  - Allows for interactive effects
  - More subtle, more realistic

- Economist models (no technical knowledge)
PF Example

- One of the advantages of the “economist” models is that they make calculations easy. This is good for examples, even if not as realistic as Technical Cost Models (next)

- Thus: Output = 2 M^{0.4} N^{0.8}

- Let’s see what this looks like...

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PF Example -- Calculation

<table>
<thead>
<tr>
<th>M</th>
<th>N</th>
<th>Output</th>
<th>N VARIABLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>31.70</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>M</td>
<td>10</td>
<td>18.21</td>
<td>31.70</td>
</tr>
<tr>
<td>20</td>
<td>0</td>
<td>24.02</td>
<td>41.83</td>
</tr>
<tr>
<td>VARIABLE</td>
<td>30</td>
<td>0.00</td>
<td>28.25</td>
</tr>
<tr>
<td>40</td>
<td>0.00</td>
<td>31.70</td>
<td>55.19</td>
</tr>
<tr>
<td>50</td>
<td>0.00</td>
<td>34.66</td>
<td>60.34</td>
</tr>
</tbody>
</table>

The formula in Excel to calculate the output is: = 2((power(b7,0.4))*(power(c7,0.8))

We calculate output for many values of the variables using a 2-way Data Table
Mathematical Representation -- Inductive

- "Engineering models" of PF
- Analytic expressions
  - Rarely applicable: manufacturing is inherently discontinuous
  - Exceptions: process exists in force field, for example transport in fluid, river
- Detailed simulation, Technical Cost Model
  - Generally applicable
  - Requires research, data, effort
  - Wave of future -- not yet standard practice

Cooling Time, Part Weight, Cycle Time Correlation (MIT MSL, Dr. Field)
PF: Characteristics

- Isoquants
- Marginal Products
- Marginal Rates of Substitution
- Returns to Scale
- Possible Convexity of Feasible Region

Characteristic: Isoquants

- Isoquant is the Locus (contour) of equal product on production function

Graph:

Production Function Surface

xi

Xj

Isoquant

Projection

yi
Important Implication of Isoquants

- Many designs are technically efficient
  - All points on isoquant are technically efficient
  - no technical basis for choice among them
  - Example:
    * little land, much steel => tall building
    * more land, less steel => low building

- Best System Design depends on Economics
- Values are decisive!

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Isoquant Example -- Calculation

For any given output, we can calculate the M value as a function of the N value. Thus:
for output = 20, the formula is:

\[ M = \text{power}(10, \frac{2.50}{\text{power}(c7,2)}) \]

A 1-way data table calculates the (M,N) combinations that constitute the isoquant

<table>
<thead>
<tr>
<th>M for OUTPUT= 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>c7=10</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>9</td>
</tr>
<tr>
<td>11</td>
</tr>
<tr>
<td>13</td>
</tr>
<tr>
<td>15</td>
</tr>
</tbody>
</table>
Isoquant Example -- Graph

Characteristic: Marginal Products

- Marginal Product is the change in output as only one resource changes

\[ MP_i = \frac{\partial Y}{\partial X_i} \]

- Graph:
Diminishing Marginal Products

- **Math:**
  
  \[
  Y = a_0 X_1^{a_1} \ldots X_i^{a_i} \ldots X_n^{a_n}
  \]
  
  \[
  \frac{\partial Y}{\partial X_i} = \frac{a_i}{X_i} Y = f(X_i^{a_i-1})
  \]

  Diminishing Marginal Product if \( a_i < 1.0 \)

- **“Law” of Diminishing Marginal Products**
  - Commonly observed -- but not necessary
  - “Critical Mass” phenomenon => increasing marginal products

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**MP Example -- Calculations**

<table>
<thead>
<tr>
<th>M</th>
<th>1.53</th>
<th>3.15</th>
<th>2.32</th>
<th>1.90</th>
<th>1.63</th>
<th>1.45</th>
<th>1.31</th>
<th>1.20</th>
</tr>
</thead>
<tbody>
<tr>
<td>C7=10</td>
<td>1.53</td>
<td>3.15</td>
<td>2.32</td>
<td>1.90</td>
<td>1.63</td>
<td>1.45</td>
<td>1.31</td>
<td>1.20</td>
</tr>
</tbody>
</table>

The formula for the marginal product is

\[= \left( \frac{0.4}{b7} \right) \cdot (2) \cdot (\text{power}(b7,0.4)) \cdot \text{power}(12.65,0.8)\]

Note that the Marginal Product is conditional on the change in only one variable (in this case M). All other variables are fixed (in this case N=12.65).

Obviously, the Marginal Product depends on the "cut" of the production function you take.
MP Example -- Graph

Characteristic: Marginal Rate of Substitution

- Marginal Rate of Substitution is the Rate at which one resource must substitute for another so that product is constant

- Graph:
Marginal Rate of Substitution (cont’d)

- Math:
  \[ \text{since } \Delta X_i \text{MP}_i + \Delta X_j \text{MP}_j = 0 \]
  (no change in product)
  then \( MRS_{ij} = \frac{\Delta X_j}{\Delta X_i} \)
  
  \[ = - \frac{\text{MP}_i}{\text{MP}_j} = - \frac{\left(\frac{a_i}{X_i}\right) Y}{\left(\frac{a_j}{X_j}\right) Y} \]
  
  \[ = - \left(\frac{a_i}{a_j}\right) \frac{X_j}{X_i} \]

- \( MRS \) is “slope” of isoquant
  - It is negative
  - Loss in 1 dimension made up by gain in other

MRS Example

- For our example PF: Output = 2 M \( ^{0.4} \) N \( ^{0.8} \)

- \( a_M = 0.4 \); \( a_N = 0.8 \)

- At a specific point, say M = 5, N = 12.65

- \( MRS = - \left(\frac{0.4}{0.8}\right) \left(\frac{12.65}{5}\right) = -1.265 \)

- At that point, it takes ~ 5/4 times as much M as N to get the same change in output
Characteristic: Returns to Scale

- Returns to Scale is the Ratio of rate of change in $Y$ to rate of change in ALL $X$ (each $X_i$ changes by same factor)

- Graph:
  - Directions in which the rate of change in output is measured for MP and RTS

\[
\text{RTS} = \frac{\sum a_i - 1}{s}
\]

Returns to Scale (cont’d)

- Math:
  \[
  Y' = a_0 \prod X_i^{a_i} \\
  Y'' = a_0 \prod (sX_i)^{a_i} = Y'(s)^{\sum a_i} \quad \text{all inputs increase by s}
  \]

\[
\text{RTS} = \frac{Y''/Y'}{s} = s^{(\sum a_i - 1)}
\]

- \[
Y''/Y' = \% \text{ increase in } Y
\]
  - if $Y''/Y' > s$ => Increasing RTS

Increasing returns to scale (IRTS) if $\sum a_i > 1.0$
Increasing RTS Example

- The PF is: \( \text{Output} = 2 \, M^{0.4} \, N^{0.8} \)
  - Thus \( \sum a_i = 0.4 + 0.8 = 1.2 > 1.0 \)
  - So the PF has Increasing Returns to Scale
  - Compare outputs for (5,10), (10,20), (20,40)

<table>
<thead>
<tr>
<th>N VARIABLE</th>
<th>10</th>
<th>31.70</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00</td>
<td>0.00</td>
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<td>0.00</td>
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<tr>
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<td>60.34</td>
<td>83.46</td>
<td>105.06</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Importance of Increasing RTS

- Increasing RTS means that bigger units are more productive than small ones

- IRTS \( \Rightarrow \) concentration of production into larger units

- Examples:
  - Generation of Electric power
  - Chemical, pharmaceutical processes
Practical Occurrence of IRTS

- Frequent!
- Generally where
  - Product = \( f \) (volume) and
  - Resources = \( f \) (surface)
- Example:
  - ships, aircraft, rockets
  - pipelines, cables
  - chemical plants
  - etc.

Characteristic: Convexity of Feasible Region

- A region is convex if it has no “reentrant” corners

- Graph:
  - CONVEX
  - NOT CONVEX
Informal Test for Convexity of Feasible Region (cont’d)

- Math: If A, B are two vectors to any 2 points in region

  \[
  \text{Convex if all} \\
  T = KA + (1-K)B \quad 0 \leq K \leq 1 \\
  \text{entirely in region}
  \]

Convexity of Feasible Region for Production Function

- Feasible region of Production function is convex if no reentrant corners

- Convexity => Easier Optimization
  - by linear programming (discussed later)
Test for Convexity of Feasible Region of Production Function

- Test for Convexity: Given $A, B$ on PF
  If $T = KA + (1-K)B$ \[ 0 \leq K \leq 1 \]
  Convex if all $T$ in region

- For Cobb-Douglas, the test is if:
  all $a_i \leq 1.0$ and $\Sigma a_i \leq 1.0$

Convexity Test Example

- Example PF has Diminishing MP, so in the MP direction it looks like left side
- But: it has IRTS, like bottom of right side
- Feasible Region is not convex
Summary

- Production models are the way to describe technically efficient systems

- Important characteristics
  - Isoquants, Marginal products, Marginal rates of Substitution, Returns to scale, possible convexity

- Two ways to represent
  - Economist formulas
  - Technical models (generally more accurate)