Black-Scholes Valuation

Richard de Neufville
Professor of Engineering Systems and of Civil and Environmental Engineering
MIT

Outline

- Background

- The Formula
  - Applicability; Interpretation; Intuition about form
  - Derivation principles

- Stochastic processes background
  - Random walk; Wiener Process (Brownian motion)
  - Ito Process; Geometric Brownian Motion

- Derivation background

- Applicability of this material to design
Meaning of “options analysis”

- Need to clarify the meaning of this term

- Methods presented for valuing options so far (lattice, etc) are all analyzing options. In that sense, they all constitute “options analysis”

- HOWEVER, in most literature “options analysis” means specific methods – based on replicating portfolios and random probability – epitomized by Black-Scholes

Keep this distinction in mind!

Background

- Development of “Options Analysis” Recent

- Depends on insights, solutions of
  - Black and Scholes ; Merton
  - Cox, Ross, Rubinstein

- This work has had tremendous impact
  - Development of huge markets for financial options, options “on” products (example: electric power)

- Presentation on options needs to discuss this – although much not applicable to engineering systems design
Key Papers and Events

- **Foundation papers:**

- **Events**
  - “Real Options” MIT Prof Myers ~ 1990
  - Nobel Prize in 1997 to Merton and Scholes (Black had died and was no longer eligible)

---

Black-Scholes Options Pricing Formula

\[ C = S \times N(d_1) - [K \times e^{-rt} \times N(d_2)] \]

It applies in a very special situation:
- a *European* call
- on a *non-dividend paying* asset

“European” \( \equiv \) only usable on a specific date
“American” \( \equiv \) usable any time in a period (usual situation for options “in” systems)

“no dividends” -- so asset does not change over period
Black-Scholes Formula -- Terms

\[ C = S \cdot N(d_1) - [K \cdot e^{-rt} \cdot N(d_2)] \]

- \( S, K \) = current price, strike price of asset
- \( r \) = Rf = risk-free rate of interest
- \( t \) = time to expiration
- \( \sigma \) = standard deviation of returns on asset
- \( N(x) \) = cumulative pdf up to \( x \) of normal distribution with average = 0, standard deviation = 1

\[
\begin{align*}
    d_1 &= \frac{\ln \left( \frac{S}{K} \right) + \left( r + 0.5 \sigma^2 \right) t}{\sigma \sqrt{t}} \\
    d_2 &= d_1 - (\sigma \sqrt{t})
\end{align*}
\]

Black-Scholes Formula -- Intuition

C = S \cdot N(d_1) - [K \cdot e^{-rt} \cdot N(d_2)]

Note that, since \( N(x) < 1.0 \), the B-S formula expresses option value, \( C \), as
- a fraction of the asset price, \( S \), less
- a fraction of discounted amount, \( (K \cdot e^{-rt}) \)

These are elements needed to create a replicating portfolio (see “Arbitrage-enforced pricing” slides). Indeed, B-S embodies this principle with a continuous pdf.
Black-S Formula – Derivation Principles

- Formula is a solution to a “Stochastic Differential Equation” (or SDE) that defines movement of value of option over time

- SDE’s defined by Ito from Japan
  - the general form known as an “Ito Process”

- Specific form of equation solved
  - embodies principle of replicating portfolio
  - makes specific assumptions about nature of movement value in a competitive market

- These ideas discussed next

Random Walks

- A “Standardized Normal Random variable”, e(t)
  - It is a “Normal” distribution (bell-shaped)
  - Mean ≡ 0 ; Standard deviation ≡ 1

- A “random walk” is a process defined by
  - $z(t + 1) = z(t) + e(t) (\Delta t)^{0.5}$

- Difference between 2 periods: $z(tk) - z(tj)$
  - Expected value = 0 ; Variance = $tk - tj$
  - Differences for non-overlapping periods are uncorrelated

- This is a random process
Wiener Process

- This is result of “random walk” as $\Delta t \to 0$

- Formally: $z(t + 1) = z(t) + e(t) (\Delta t)^{0.5}$
- Becomes: $dz = e(t) (dt)^{0.5}$

- Also known as “Brownian Motion” in science or “white noise” in engineering

- As for random walk:
  - $z(t) - z(s)$ is a normal random variable
  - for any 4 times $t_1 < t_2 < t_3 < t_4$ $[z(t_1) - z(t_2)]$ and $[z(t_3) - z(t_4)]$ are uncorrelated

Generalized Wiener process

- An extension of Brownian motion
  $dx(t) = a \, dt + b \, dz$

- In short, it
  - represents a growth trend: $a \, dt$
  - Plus white noise: $b \, dz$

- It can be solved: $x(t) = x(0) + a \, t + b \, z(t)$

- This is similar to what lattice represents – but see next slides
Ito Process

- A further extension...

- Basic Eqn: \( dx(t) = a \, dt + b \, dz \)
- Becomes: \( dx(t) = a(x, t) \, dt + b(x, t) \, dz \)

- In short, coefficients can change with time

- This is a “stochastic differential equation”
  - Stochastic because it varies randomly with time

Application to Asset Prices -- GBM

- Asset prices assumed to fluctuate around a multiplicative growth trend
  - For example \( S_0 \rightarrow u \, (S_0) \) or \( d \, (S_0) \)

- The continuous version of this is:
  \[
  d \, [\ln S(t)] = v \, dt + \sigma \, dz
  \]
- This is a generalized Wiener process

- With solution: \( \ln S(t) = \ln S_0 + vt + \sigma \, z(t) \)
- This is: Geometric Brownian Motion (GBM)
Standard Itô form

- This is the solution for $S(t)$.

- $\frac{d S(t)}{S(t)} = (v + 0.5 \sigma^2) dt + \sigma dz$
  - Solution not obvious -- A special case of Itô’s lemma

- Interpret this as saying that:
  - Relative change of asset value, $\frac{d S(t)}{S(t)}$ = a trend (constant) $dt$
  - $\mu = v + 0.5 \sigma^2$ [0.5 $\sigma^2$ is correction factor]

Ito’s lemma

- If: $x(t)$ is defined by Itô process
  \[ dx(t) = a(x, t) dt + b(x, t) dz \]

- And $y(t) = F(x, t)$

- Then:
  \[ dy(t) = \left[ \left( \frac{\delta F}{\delta x} \right) a + \frac{\delta F}{\delta t} + 0.5 \left( \frac{\delta^2 F}{\delta x^2} b^2 \right) \right] dt + \left( \frac{\delta F}{\delta x} b \right) dz \]

- In words: given a “derivative” of an asset, $F(x, t)$, we have an equation defining value of derivative
Derivation Background

- Suppose value of Asset is random process:
  \[ dS(t) = \mu S \, dt + \sigma S \, dz \]
- And that we can borrow money at rate \( r \)

- The price of a derivative (an option) \( f(S,t) \) of this asset satisfies the Black-Scholes equation:
  \[
  \frac{\partial f}{\partial t} + \left( \frac{\partial f}{\partial S} \right) r S + \left( \frac{\partial^2 f}{\partial S^2} \right) \sigma^2 S^2 = rf
  \]

- Unless this property is met – arbitrage opportunity exists
- Solution to equation defines price of derivative

Black-Scholes Formula as Solution

- It is the solution to the Black-Scholes equation

- Meeting the boundary conditions:
  - It is a call
  - There is only 1 exercise time (European option)
  - The asset “pays no dividends” – that is, gives off no intermediate benefit (mines or oil wells generate ‘dividends’ in exploitation, so B-S does not apply)

- Development a brilliant piece of work

- Why do we care?
Why does this matter?

- Development of Formula showed the way for financial analysts

- Essentially “no” other significant closed form solutions...

- But solutions worked out numerically through lattice (and more sophisticated) analyses

- Led to immense development of use of all kinds of “derivatives”

---

Why does this matter TO US?

- What does B-S mean to designers of technological systems?

- Important to understand the assumptions behind Black-Scholes equation and approach

- Extent these assumptions are applicable to us, determines the applicability of the approach

- Much research needed to
  - address this issue
  - Develop alternative approaches to valuing flexibility
Price Assumption

- B-S approach assumes that we have a “price” for the Asset

- When is this true
  – System produces a commodity (oil, copper) that has quoted prices set by world market

- When this may be true
  – System produces goods (cars, CDs) that lead to revenues and thus value – HOWEVER, prices for depend on both design and management decisions

- When this is not true
  – System delivers services that are not marketable, for example, national defense…

Replicating Portfolio Assumption

- B-S analysis assumes that it is possible to set up replicating portfolio for the asset

- When is this true
  – Product is a commodity

- When this might assumed to be true
  – Even if market does not exist, we might assume that a reasonable approximation might be constructed (using shares in company instead of product price)

- When this is probably a stretch too far
  – Privately owned concern, whose owners may not be concerned with arbitrage against them, and thus may want to use actual probabilities…
Volatility Assumption

- B-S approach assumes that we can determine volatility of asset price

- When this is true
  - There is an established market with a long history of trades that generates good statistics

- When this is questionable
  - The market is not observable (for example, because data are privately held or negotiated)
  - Assets are unique (a prestige or special purpose building or special location)

- When this is not true
  - New technology or enterprise with no data

Duration Assumption

- B-S approach assumes volatility of asset price is stable over duration of option

- When this is true
  - Short-term options (3 months, a year?) in a stable industry or activity

- When this is questionable
  - Industries that are in transition – technologically, in structure, in regulation – such as communications

- When this is not true
  - Long-duration projects in which – major changes in states of markets, regulations or technologies are highly uncertain (Exactly where we want flexibility!)
Take-Away from this Discussion

- In many situations the basic premises of “options analysis” – as understood in finance – are unlikely to apply to the design and management of engineering systems.

- Yet these systems, typically being long-life, are likely to be especially uncertain – and thus most in need of flexibility – of “real options”.

- We thus need to develop pragmatic ways to value options for engineering systems.

TOPIC OF SUBSEQUENT PRESENTATIONS

Summary

- Black-Scholes formula elegant and historically most important.

- Its derivation based on some fundamental developments in Stochastic processes – Random walk; Wiener and Ito Processes; GBM.

- Underlying assumptions limit use of approach – Price; Replicating Portfolio; Volatility; Duration.

- Developing useful, effective approaches for design is an urgent, important task.