Arbitrage Enforced Valuation

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Outline

• Replicating Portfolio – key concept
• Motivating Example ➔ Value Independent of Objective probabilities!!
• Arbitrage Enforced Pricing
• Application to Binomial Lattice Analysis
• Risk Neutral “probabilities”
Definition: Replicating Portfolio

- A “replicating portfolio” is..
- A set of assets (a portfolio) that has same payoffs -- replicates – payoffs of option

Example for a “call” option
- If asset value goes up, exercise option and option profit = portfolio profit
- If asset value down, do not exercise option and option value = 0 = portfolio value

Note: Replicating Option is not obvious, … must be constructed carefully

Use of Replicating Portfolio

- Why is a Replicating Portfolio Useful?
- … Because it may be easier to value portfolio than option

- Since by construction the portfolio exactly replicates payoffs of option
- Thus: value of portfolio \( \equiv \) value of option
- So we can value option as sum of values of replicating portfolio
  … as example will show
Basis for Replicating Portfolio -- Call

Think about what a call option provides...

- It enables owner to get possible increased value of asset
  - If exercised, call option results in asset ownership

- However, it provides this benefit without much money! Payment for asset delayed until option is exercised
  - Ability to delay payment is equivalent to a loan

- Therefore: A Call option is like buying asset with borrowed money

Basis for Replicating Portfolio -- Put

Argument is similar for a put...

- Put enables owner to avoid possible loss in value of asset
  - If exercised, put option results in sale of asset

- However, it provides this benefit without early commitment! Delivery of asset delayed until option is exercised
  - Ability to delay delivery is equivalent to a loan

- Therefore: A Put option is like getting cash (i.e., selling asset) with borrowed asset
Example will illustrate

- Explanation for replicating portfolio (e.g., call as “buying asset with a loan”) ... is not obvious
- Example will help, but you will need to think about this to develop intuition
- Bear with the development!

Motivating Example

- Valuation of an example simple option has fundamentally important lessons
- Key idea is possibility of replicating option payoffs using a portfolio of other assets
  - Since option and portfolio have same payoffs,
  - The value of option = value of portfolio
- Surprisingly, when replication possible:
  value of option does NOT depend on probability of payoffs!
Motivating Example – Generality

- The following example illustrates how a replicating portfolio works in general
- The example makes a specific assumption about how the value of the asset moves...
- The principle used to replicate the option does not depend on this assumption, it can be applied to any assumption
- Once you make assumption about how the asset moves, it is possible to create a replicating portfolio

A Simple 1-Period Option

- Asset has a Current price, \( S_0 = \$100 \)
- Price at end of period either
  \( S_{\text{DOWN}} = \$80 \) or \( S_{\text{UP}} = \$125 \)
- One-period call option to buy asset at Strike price, \( K = \$110 \)
- What is the value of this option?
- More precisely, what is the price, \( C \), that we should pay for this option?
**Graphically…**

- Call Option on $S$, $S$ currently worth $100 = S_0$
- strike price $= K = 110$
- possible values of $S$: $S_{DOWN} = 80$; $S_{UP} = 125$

![Graphical Illustration]

**Call Option Cost and Payoffs**

- Fair Cost of Option, $C$, is its value. This is what we want to determine

- If at end of period
  - asset price $> $ strike price: option payoff $= S - K$
  - asset price $< $ strike price: option payoff $= 0$

<table>
<thead>
<tr>
<th>Asset Price</th>
<th>Start</th>
<th>End</th>
<th>End</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy Call</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Strike = 110</td>
<td>$- C$</td>
<td>0</td>
<td>$(125 - 110) = 15$</td>
</tr>
</tbody>
</table>
Replicating Portfolio Cost and Payoffs

- Replicating Portfolio consists of:
  - Asset bought at beginning of period
  - Financed in part by borrowed money

- Amounts of Asset bought and money borrowed arranged so that payoffs equal those of option
- Specifically, need to have asset and loan payment to net out as follows:
  - If $S > K$, want net = positive return
  - If $S < K$, want net = 0

Note: This is first crucial point of arrangement!

Creating the replicating portfolio

- This description designed to show what is going on –in practice, short-cut procedure is used

- Recognize that (net value of portfolio)
  \[ = (\text{asset value} - \text{loan repayment}) \]
- To arrange that (net value of portfolio) = 0...
  - We set: (loan repayment) = $S_{\text{DOWN}} = 80$

- Note: (loan repayment)
  \[ = (\text{amount borrowed}) + (\text{interest for period}) \]
  \[ = (\text{amount borrowed}) (1 + r) \]
  \[ (\text{Amount borrowed}) = \frac{80}{1+ r} \]
- The situation has 3 elements

- Value of Asset is up or down
- Value of call option is up or down
- Value of loan rises by Rf (no risk) to
  \[ R = 1 + R_f \]
  (for 1 year)

Table of Portfolio Cost and Payoffs

<table>
<thead>
<tr>
<th></th>
<th>Start</th>
<th>End</th>
<th>End</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset price</td>
<td>100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Buy Asset</td>
<td>-100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Borrow Money</td>
<td>80/(1+r)</td>
<td>-80</td>
<td>-80</td>
</tr>
<tr>
<td>Net</td>
<td>-100 + 80/(1+r)</td>
<td>0</td>
<td>45</td>
</tr>
</tbody>
</table>

Observe: net of portfolio is like payoff of option
Comparing Costs and Payoffs of Option and Replicating Portfolio

- If $S < K$, both payoffs automatically = 0 by design
- If $S > K$, call payoff is a multiple of portfolio payoff (in this case, ratio is 1:3)
- Thus: payoffs of call = payoffs of $\left(\frac{1}{3}\right)$ portfolio
- Also: Net cost of portfolio = - [asset cost – loan]

<table>
<thead>
<tr>
<th>Period</th>
<th>Start</th>
<th>End</th>
<th>End</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset Price</td>
<td>100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Buy Call</td>
<td>- C</td>
<td>0</td>
<td>$125-110 = 15$</td>
</tr>
<tr>
<td>Buy Asset And Borrow</td>
<td>-100 + 80/(1 + r)</td>
<td>0</td>
<td>45</td>
</tr>
</tbody>
</table>

Value of Option (1)

- Value of Option = $(1/3)$ (Value of Portfolio)
- $C = (1/3)[ -100 + 80/ (1 + r)]$
  
  … calculated at appropriate $r$ -- What is that?
Implications of: Option = Portfolio

- Crucial observation:
  - Seller of option can counter-balance this with a portfolio of equal value, and thus arrange it so cannot lose!

- Such a no-risk situation is known as ARBITRAGE

- Since arbitrage has no risk, appropriate DISCOUNT RATE = Rf = RISK FREE RATE!

- This is second crucial point of arrangement

Value of Option (2)

- The appropriate value of option is thus
- assuming Rf = 10% (for easy calculation)
- \(- 3C = \frac{-100 + 80}{1 + Rf} = -$27.27\)
- \(C = $9.09\)

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<tr>
<th>Period</th>
<th>Start</th>
<th>End</th>
<th>End</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset Price</td>
<td>100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Buy 3 Calls</td>
<td>-3C</td>
<td>0</td>
<td>45</td>
</tr>
<tr>
<td>Buy Asset And Borrow</td>
<td>-100 + 80/(1 + r) = -27.27</td>
<td>0</td>
<td>45</td>
</tr>
</tbody>
</table>
Value independent of actual probability!

- Nowhere in calculation of the option value is there any statement about actual probability that high or low payoffs (125 or 80) occur.

- In situations as described, actual probabilities do not matter!
- Very surprising, since options deal with uncertainties
- A remarkable, counter-intuitive result
- What matters is the RANGE of payoffs

No knowledge or need for pdf

- Recognize that all the only thing we needed to know about the asset was the possible set of outcomes at end of stage

- The “asset” is like a black box – we know
  - what comes in (in this case S = 100)
  - What comes out (in this case, 80 or 125)

- Nowhere do we know anything about pdf

- With arbitrage-enforced pricing, we are not in Expected Value world (in terms of frequency)
Arbitrage-Enforced Pricing -- Concept

- Possibility of setting up a replicating portfolio to balance option absolutely defines value (and thus market price) of option

- This because this permits market pressures to drive the price of option to a specific value

- This is known as “Arbitrage-Enforced Pricing”

- THIS IS CRUCIAL INSIGHT!!!
- It underlies all of options analysis in finance (note implied restriction)

Arbitrage-Enforced Pricing -- Mechanism

- How does “Arbitrage - Enforced Pricing” work?

- If you are willing to buy option for \( C^* > C \), the price defined by portfolio using risk-free rate than someone could sell you options and be sure to make money -- until you lower price to \( C \)

- Conversely, if you would sell option for \( C^* < C \), then someone could buy them and make money until option price = \( C \)

- Thus: \( C \) is price that must prevail
Arbitrage-Enforced Pricing -- Assumptions

- When does “Arbitrage-Enforced Pricing” work?

- Note key assumptions:
  - Ability to create a “replicating portfolio”
    - This is possible for financially traded assets
    - May not be possible for technical systems (for example, for a call on extra capacity, or use of spare tire for car)
  - Ability to conduct arbitrage by buying or selling options and replicating portfolios
    - There may be no market for option or portfolio, so price of option is not meaningful, and market pressure cannot be exercised
  - If Assumptions not met, concept dubious

Arbitrage-Enforced Pricing -- Applicability

- For options on traded assets (stocks, foreign exchange, fuel, etc.), it is fair to assume that conditions for arbitrage-enforced pricing exist
- Arbitrage-enforced pricing thus a fundamental part of traditional “options analysis”

- For real options, “on” and “in” technical systems, the necessary assumptions may not hold
- It is an open issue whether and when arbitrage-enforced pricing should be used

- In any case: You need to know about it!
Basis for Options analysis

- The valuation of this very simple option has fundamentally important lessons

- Surprisingly, when replication possible:
  value of option does NOT depend on probability of payoffs!

  - Contrary to intuition associated with probabilistic nature
  - This surprising insight is basis for options analysis

Application to Binomial Lattice

- How does arbitrage-enforced pricing apply to the binomial lattice?

- It replaces actual binomial probabilities (as defined by growth rate, \( v \), and standard deviation, \( \sigma \)) by a relative weights derived from replicating portfolio

- These relative weights reflect the proportion ratio of asset and loan (as in example) – but look like probabilities: they are the risk-neutral “probabilities”
Single Period Binomial Model Set-up

- Apply to generalized form of example

Value of Asset is up or down

\[ S \quad \frac{\text{up}}{\text{down}} \]

Value of call option is up or down

\[ C := \begin{cases} \max(S_u - K, 0) = C_u \\ \max(S_d - K, 0) = C_d \end{cases} \]

Value of loan rises by Rf (no risk) to

\[ R = 1 + R_f \] (for 1 year)

Single Period Binomial Model Solution

- The issue is to find what proportion of asset and loan to have to establish replicating portfolio

- Set: asset share = “x” loan share = “y”
- then solve:
- \[ x u S + y R = C_u \] and \[ x d S + y R = C_d \]

\[ \Rightarrow x = \frac{(C_u - C_d)}{S(u - d)} \]
\[ \Rightarrow y = \frac{(1/R) [u C_d - d C_u]}{(u - d)} \]

Portfolio Value = Option Price

\[ = \frac{[(R - d) C_u + (u - R) C_d]}{R(u-d)} \]
Application to Example

For Example Problem:
R = 1 + Rf = 1.1 \text{ (Rf assumed = 10\% for simplicity)}
Cu = \text{value of option in up state} = 15
Cd = \text{value of option in down state} = 0
u = \text{ratio of up movement of } S = 1.25
d = \text{ratio of down movement of } S = 0.8

Portfolio Value = Option Price
\begin{align*}
= & \frac{[\text{R} - \text{d}]\text{Cu} + \text{(u - R)}\text{Cd}}{\text{R}(\text{u} - \text{d})} \\
= & \frac{[1.1 \cdot 0.8 \cdot (15) + (1.25 - 1.1)(0)]}{1.1(1.25 - 0.8)} \\
= & \frac{0.3(15)}{1.1(0.45)} = 10 / 1.1 \\
= & 9.09 \text{ as before}
\end{align*}

Reformulation of Binomial Formulation

Option Price = \frac{[\text{R} - \text{d}]\text{Cu} + (\text{u - R})\text{Cd}}{\text{R}(\text{u} - \text{d})}

\begin{itemize}
  \item We simplify writing of formula by substituting a single variable for a complex one:
  \begin{align*}
  \text{“q”} & \equiv \frac{\text{R} - \text{d}}{\text{u} - \text{d}} \\
  \text{Option Price} & = \frac{[\text{R} - \text{d}]\text{Cu} + (\text{u - R})\text{Cd}}{\text{R}(\text{u} - \text{d})} \\
  & = \frac{1}{\text{R}}[\left\{((\text{R-d})/(\text{u-d}))\text{Cu} + (\text{u-R})/(\text{u-d})\text{Cd}\right\}] \\
  & = \frac{1}{\text{R}} [q\text{Cu} + (1-q) \text{Cd}]
  \end{align*}
  \item Option Value is weighted average of q, 1-q
\end{itemize}
q factor = risk-neutral “probability”

- Option Price = $(1/R) \left[ qCu + (1-q) Cd \right]$

- This leads to an extraordinary interpretation! Value of option = “expected value” with binomial probabilities q and (1 - q)
- These called: “risk-neutral probabilities”
- Yet “q” defined by spread: $q \equiv (R - d) / (u - d)$ actual probabilities do not enter into calculation!

\[
\begin{array}{c}
\text{C} & q & \text{Max}(Su - K, 0) = Cu \\
(1-q) & & \text{Max}(Sd - K, 0) = Cd
\end{array}
\]

Binomial Procedure using q

- “Arbitrage-enforced” pricing of options in binomial lattice proceeds as with “decision analysis” based valuation covered earlier

- Difference is that probabilities are no longer $(p, 1-p)$ but $(q, 1-q)$
- From the perspective of calculation, $(q, 1-q)$ are exactly like probabilities
- However, never observed as frequencies, etc.
- Said to be “risk-neutral”, because derived from assumption of risk-free arbitrage
Summary

- **Replicating Portfolio – key concept**
  - A combination of asset and loan
  - Designed to give same outcomes as option
  - Leads to possibility of valuation of option

- **Arbitrage Enforced Pricing**
  - Option value determined by replicating portfolio
  - Provided that assumptions hold
  - “always” for traded assets;
  - Unclear for real options

- **Risk Neutral “probabilities”**
  - Represent Arbitrage-enforced valuation in lattice