Financial Options

- Option:
  - Right (but not obligation)
  - To buy ("call") or sell ("put")
  - An underlying asset
  - At ("European") or by ("American") a certain date
  - For a specified price ("strike" or "exercise" price)
- Options are a class of financial "derivatives"—value is derived from the value of an underlying asset
- Option contract specifies:
  - Name of underlying asset (S)
  - Number of shares of the underlying asset optioned
  - Type (put or call)
  - Rule of exercise (European or American)
  - Expiration date (T)
  - Exercise price (K)

Leverage

- Options allow for a huge amount of leverage
  - Can be used by hedgers (portfolio insurance)
  - Example: Insure against drop in prices of a particular asset over a period of time by buying a put option
Example of leverage

- Invest £100
  - Current price of a share in firm A: £50
  - Current price for call option on one share in A at strike price £50 is £2.50
- Portfolio 1: 2 shares in A
- Portfolio 2: 40 call options
- Scenario 1: $S_0=£55
  - Return of portfolio 1: (110-100)/100=10%
  - Return of portfolio 2: (40*5-100)/100=100%
- Scenario 2: $S_0=£45
  - Return of portfolio 1: (90-100)/100=-10%
  - Return of portfolio 2: (40*0-100)/100=-100%

Leverage is very attractive to speculators...

http://www.aer.cs.wisc.edu/casanello/132.html

“Nick Leeson was an investment officer of Barings Bank London England. Mr. Leeson worked out of the bank’s Singapore office. Mr. Leeson was accused of losing £2.3 billion dollars as a result of a risky derivatives investment with the potential of a 27 billion gain. To be fair to Mr. Leeson, he had made other investments that had made significant gains. Nick blamed senior management in London for the debacle and they blamed Nick. It should be noted that accounting safeguards were apparently not in place in the Singapore office. After capture, Nick was held in confinement for nine months in a Frankfurt Germany prison and was eventually extradited to Singapore where the alleged crimes were committed. Nick was sentenced to 6 1/2 years by the Singapore court for forgery and cheating. The result of the £2.3 billion loss was the financial collapse of one of the world’s largest banks”

The value of an option

- Buyer of the option has to pay a price to writer of the option
- In contrast to futures
- What is a fair (i.e. market) price of an option?
- Today’s objective is to explain how options can be priced, using the so-called binomial lattice method
- Pricing of financial options is done by constructing a portfolio of traded assets that has precisely the same payoffs as the option
Replicating payoffs: Put-call parity for European options

- Portfolio 1: buy stock for $S_0$ and put at strike price $K$
  - Payoff if $S_T < K$: $K$  
  - Payoff if $S_T = K$: $S_T$
- Portfolio 2: buy call at strike price $K$ and put
  $K/(1+r)^T$ in bank account at interest $r$
  - Payoff if $S_T < K$: $K$  
  - Payoff if $S_T > K$: $S_T - K$
- Payoffs are identical!
  - Hence: value of stock plus value of put = value of call plus $K/(1+r)^T$
  - Formally: $S + P = C + K/(1+r)^T$

Let's look at gambling

- Gamble with payoffs determined by flipping a coin

```
  \[ \begin{array}{cc}
    ? & £20 \\
    1/2 & £9 \\
    1/2 & £0 \\
  \end{array} \]
```

- What's the value of Gamble 0?

Other gamble available on the same underlying uncertainty

```
  \[ \begin{array}{cc}
    £3 & £6 \\
    1/2 & £2 \\
    1/2 & £0 \\
  \end{array} \]
```

- Payoffs determined by the same coin flip as for Gamble 0
- Positive expected return of £1 is “price for risk”
- Can change stakes and payoff scales accordingly
- Gamble 0 can be interpreted as an option of buying 10 shares of Gamble 1 at strike price £4 after the flip of the coin
- What is the value of Gamble 0 in the light of the “traded” Gamble 1?
- Tradability: There is sufficient supply and demand for the price of £3
Replicating payoffs of Gamble 0

- Payoff spread of Gamble 0 is £20
- Need to invest £15 in Gamble 1 to get the same spread

£15

£30

£10

- This gamble is obviously worth £10 more than Gamble 0
- Therefore the market price for Gamble 0 is £5
- Replicating investment strategy: borrow £10, invest £15 in Gamble 1, and repay your borrowed £10 (assume no interest)
- Payoffs of investment alternative will be precisely the same as for Gamble 0
- Outlay to realize investment alternative is £5

Including risk free alternative

- Payoffs of Gamble 0 can be exactly replicated by buying 5 shares of Gamble 1 and offering (selling) 10 shares of Gamble 2
- Necessary outlay is £15: £15 = £5  That's the market price for Gamble 0

£3

1/2

£8

1/2

Gamble 1

£1

1/2

£2

1/2

Gamble 2

(risk free)

Arbitrage

- Suppose Gamble 0 is priced for less than £5
- How can you make risk free gains (arbitrage)?
  - Replicating portfolio: buy 5 shares of Gamble 1 and sell 10 shares of Gamble 2
  - Replicating portfolio has the same payoffs but costs more than Gamble 0
  - Arbitrage strategy: buy Gamble 0 and sell replicating portfolio
  - Riskless profit
- But: A rational and fully informed player will buy the replicating portfolio if Gamble 0 was available for less than £5
- Gamble 0 disappears from the market if its price is less than £5
- Similarly: Gamble 1 disappears from the market if price for Gamble 0 is more than £5
- "Co-existence" (or equilibrium) price is precisely £5
An important observation

- The equilibrium price of Gamble 0 is independent of the probabilities on the arcs!
- The reason is that all Gambles are in the replicating portfolio are based on the same underlying uncertainty
- Only the consequences (payoffs) of gambles are different
- We produce a replica of Gamble 0, using the existing gambles in the market, which produces EXACTLY the same
  consequences for each possible scenario
- Therefore the probability of occurrence of the scenarios is
  inconsequential for the price of Gamble 0
- By changing probabilities we can change the expected payoff of "stock" Gamble 1 (to any number between £2 and £6)
- This has no effect on the value of the "option" Gamble 0
- The value is only affected by the spread between the two payoffs (£6-£2) ("volatility" of the stock)

Let’s price another gamble

- What’s the value in the light of the alternative?

Solution

- Stake £2.25 on Gamble 1 gives same spread of payoffs
- New gamble is equivalent to £2.25 at stake in Gamble 1 plus £5.50 risk free in your pocket
- Therefore price is £7.75
- Notice: Need a risky (i.e. spreaded) asset to replicate the spread and a risk-free asset to shift the spread to the right level
A more complicated gamble

What's the value of this gamble in the light of the alternatives?

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Working backwards

- The fair price for upper right-hand branch was determined to be £7.75
- The fair price for the lower right-hand branch is £1.75 (invest £5.25 in Gamble 1 and borrow £3.5 by selling 3.5 shares of Gamble 2)
- Therefore the gamble is equivalent to

  1/2 £7.75
  1/2 £1.75

- This is replicated by borrowing £1.25 and investing £4.5 in Gamble 1
- Fair price of the gamble is £4.5 - £1.25 = £3.25

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Rebalancing the replicating portfolio over time

- Replicating portfolio at the beginning:
  - Buy Gamble 1 for £4.5 and sell Gamble 2 for £1.25
- If first move is upwards then we obtain £7.75 and rebalance the portfolio to
  - Invest in Gamble 1 for £2.25 and in Gamble 2 for £5.50
- If first move is downwards then we obtain £1.75 and rebalance the portfolio to
  - Invest £3.25 in Gamble 1 and sell Gamble 2 for £3.5
Intermediate Summary

- To price an option, one replicates the outcomes by combining traded assets
- The fair price of the option is then the price of the replicating portfolio which is available since the assets in the portfolio are traded in the market
- In pricing an option we need to bear in mind that the price changes over time, depending on the resolution of the uncertainties
  - Every time an uncertainty becomes resolved, the price of the option changes and the replicating portfolio needs to be rebalanced
- A simple option can be priced through a tree (called a binomial lattice), provided the uncertainty of the underlying asset has a simple tree structure

A simple model of stock prices

- Assumption: Given a stock price $S$ today, the stock will move over a short period $\Delta t$ to $uS$ (upward move) with probability $p$ and to $dS$ (downward move) with probability $(1-p)$

$$
S \xrightarrow{p} uS \quad \xrightarrow{(1-p)} dS
$$

- $u$ and $d$ are numbers with $u>d>0$, typically $u>1$ (increase in stock prices) and $d<1$ (decrease in stock prices)
- Let us see how this model develops over time…
  (see Option Valuation in a Lattice.xls)

Unfolding of stock price uncertainty

<table>
<thead>
<tr>
<th>Time</th>
<th>Stock Price</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>S</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>uS</td>
<td>p</td>
</tr>
<tr>
<td></td>
<td>dS</td>
<td>(1-p)</td>
</tr>
<tr>
<td>2</td>
<td>u(uS)</td>
<td>p^2</td>
</tr>
<tr>
<td></td>
<td>u(dS)</td>
<td>p(1-p)</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Log-normal stock prices

- Observation: The logarithm of stock prices is approximately normal for large time horizons
  - Can be shown mathematically that it tends to a normal distribution as the number of periods tends to infinity and the period length goes to zero
  - Formally: \( \ln \left( \frac{S_t}{S_0} \right) \)
    - Obviously the log return \( \ln \left( \frac{S_t}{S_0} \right) \) is a normal variable as well
    - A random variable whose logarithm is a normal are said to have a log-normal distribution
      - i.e. X is log normal if and only if \( X = e^Y \) for a normal Y
Which parameters?

- Need to determine parameters $p, u, d$ for the binomial model from market data
- Estimate $\nu = E(\ln(S_T / S_0))$, $\sigma = STD(\ln(S_T / S_0))$
- Mathematical result: Suppose the period length is $T/m$ and $n$ is the number of periods in the lattice. If we set

$$u = e^{(\nu T/m)} \quad d = \frac{1}{u} \quad p = \frac{1}{2} \left(1 + \frac{\sigma \sqrt{T/m}}{2}\right)$$

then the lattice produces approximately log-normally distributed stock prices with the given expectation and variance
- Notice that $u$ and $d$ do not depend on $\nu$

Option pricing by binomial lattices: Single period model

Replicating the option payoffs

- Invest $Ex$ in stock and $Ey$ in risk-free asset
- Stock is used to replicate the spread of the option:

$$xu = d = C_u - C_d$$

(or equivalently: $C_u = (1+q)y(xu)$)
- Replicating portfolio is therefore

$$x = (C_u - C_d)/(u-d)$$

$$y = (C_u - C_d)/(1+q)$$

- Price of the option is $Ex + Ey$ which, after some algebra, becomes

$$x+y = \frac{(C_u + (1-q)C_d)(1+r)}{q((1+r)/(1-d)) - (u-d)}$$
### Equivalent derivation

- Upwards match: \( ux(1+r)y=C_u \)
- Downwards match: \( dx(1+r)y=C_d \)
  - Two equations in two unknowns \( x,y \)
  - Solution:
    
    \[
    x = \frac{(C_u-C_d)(u-d)}{u-d} \\
    y = \frac{(C_u-xu)(1+r)}{1+r}
    \]

### Risk neutral pricing

- Pricing formula
  
  \[
  x+y = qC_u + (1-q)C_d \]

  can be interpreted as expected payoff of the option discounted at the risk-free rate \( r \), if the upward probability \( p \) of the stock process is replaced by the “risk-neutral” probability

  \[
  q = \frac{(1+r)-d}{u-d}
  \]

  - Notice that \( p \) does not occur anywhere in the pricing process and, in particular, \( q \) does not depend on \( p \!
  
  - Therefore: Options pricing is the same as doing NPV with the risk-free discount rate, provided we make a “risk-adjustment” to the stock price process
  
  - This approach is called risk-neutral pricing
  
  - However: What’s the intuition behind this?

### Multi-period lattices

- Value of the option after two periods is
  
  - Two upward moves: max\((uS-K,0)\)
  
  - One upward, one downward move: max\((udS-K,0)\)
  
  - Two downward moves: max\((d^2S-K,0)\)

  - Recall risk-neutral probability \( q = (1+r-d)/(u-d) \)
  
  - Work backwards through the lattice
  
  - Repeat single-period risk-free discounting at every node of the lattice, starting from the final period
Example

- Data:
  - Stock price is currently £62, Estimated logarithm of return \( e = 0.2 \) over a year \((T=1)\)
  - European call option over 5 months at strike price \( K = 60 \)
  - Risk-free rate is 10%, compounded monthly \((r=0.1/12, T/m=1/12)\)
- Conversion of this information to lattice parameters:
  \[ u = e^{T/m} = 1.059 \quad d = e^{-T/m} = 0.944 \]
- Risk-neutral probability: \( q = \frac{(1+r)-d}{u-d} = 0.559 \)

The lattice

<table>
<thead>
<tr>
<th>Stock price</th>
<th>Option value</th>
<th>Stock price</th>
<th>Option value</th>
<th>Stock price</th>
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<td>£73.63</td>
<td>£14.62</td>
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<td>£46.44</td>
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<td>£42.98</td>
<td>£0.00</td>
</tr>
</tbody>
</table>

Working backwards through the tree...

- Discounted expected payoff at risk neutral probability and risk-free discount rate

Page 11
Conclusions

- Options are priced by constructing a portfolio of traded assets which exactly replicates the payoffs of the option (for each possible scenario)
  - Portfolio needs to be rebalanced over time to take account of the change of option value over time
- The replication can be performed in a binomial lattice
  - Based on binary movements over short time intervals
- Process is equivalent to NPV calculation for an adjusted stock price process (risk neutral valuation)
- This presentation was to some extend based on chapters 11-13 of Luenberger: *Investment Science*

Homework

- Do Examples 12.7 (page 337) and 12.10 (page 341) and Exercise 8 (page 348) in Luenberger's book
- Try to understand the assumptions that underlie the financial options valuation process
- Read the Antamina case
  - Formulate the problem in an options framework
  - Are the assumptions for financial option pricing appropriate in this case? Be prepared to defend your views in class