

## Chapter 6

### 6.1 Sensitivity Exercise I

a) See Figure S6.1

b)  $X^* = (11, 5)$        $Z = 1$

c)  $SP_5 = 0$

d)  $SP_5 = 0$  for  $b_5 \geq -26$ , where constraint #5 starts to bind.

e) To find  $SP_5$  for  $b_5 < -26$ : let  $\Delta = -4$ , so constraint #5 is  
 $-X_1 - 3X_2 \leq -30$ .  $SP_5 = (\Delta Z)/\Delta b_5 = 0.25$ .

Note that constraint #1 is no longer binding: it has been replaced by constraint 5.

f) See Figure S6.2.  $SP_5$  will remain constant at 0.25 for  $-26 > b_5 > -42$ . At this point the optimal solution becomes (15, 9),  $Z = 13$ , constraint #4 ceases to be binding and constraint #2 becomes binding.  
For  $-42 > b_5 > -50$ ,  $SP_5 = 11/4$ .

For  $b_5 < -50$ , the feasible region disappears and the problem is insoluble.  $SP_5 = \infty$

### 6.2 Sensitivity Exercise II

a)  $Z = 21$

b)  $X_1 = 6, X_2 = 0, X_3 = 3$

c) Using 1 unit  $X_2$  implies giving up 2 units  $X_3$  and 3 units  $X_1$ .  
 $\Delta Z = +2 - 10 - 3 = 11$  ;      OC of  $X_2 = 11$

d)  $W_1 = 1, W_2 = 5, W_3 = 0$

e)  $SV_1 = 0, SV_2 = 0, SV_3 = 6$

### 6.3 Sensitivity Exercise III

a)  $Z = 21$

b)  $X_1 = 1, X_2 = 5, X_3 = 0$

c) Using 1 unit of  $X_1$  implies giving up 1 unit each of  $X_1$  and  $X_2$ .  
 $\Delta Z = +5 - 6 - 3 = 6$  ;      OC of  $X_3 = 6$

d)  $W_1 = 6, W_2 = 0, W_3 = 3$

e)  $SV_1 = 0, SV_2 = 11, SV_3 = 0$

#### 6.4 Graphic Analysis

- a) See Figure S6.3.
- b) For  $(X_1, X_2)$  the basic feasible solutions are:  $(0,0)$ ,  $(0,3)$ , and  $(4,0)$ .  $(4,0)$  is optimal with  $Z = 12$
- c) The shadow price of the first constraint is 1. The shadow price of the second constraint is zero because it is not binding. The opportunity cost of  $X_2$  is 2. Using 1 unit of  $X_2$  requires  $4/3$  less units of  $X_1$

#### 6.5 Primal-Dual

- a)  $Z^* = 27$
- b)  $X^* = (3,0,1.5)$
- c) Introducing 1 unit of  $X_2$  increases  $Z$  by 12. It also satisfies constraint #3 without using any  $X_1$ , so the objective function decreases by 3  $(6) = 18$ . But 1.5 more units of  $X_3$  are needed to satisfy constraint #1:  
 $\Delta Z = (6)(1.5) = 9$        $OC_2 = 12 - 18 + 9 = 3$   
 $OC_1 = OC_3 = 0$  (since they are optimal), so  $OC^* = (0,3,0)$
- d) If  $b_1$  is raised by one,  $X_3$  would increase  $1/2$  and  $\Delta Z = 3$   
If  $b_2$  is raised by one, there is no change, since the left hand side is already 4.5.  $\Delta Z = 0$   
If  $b_3$  is raised by one, an extra unit of  $X_1$  is required, but that allows  $1/2$  unit less of  $X_3$  to be used to satisfy constraint #1.  $\Delta Z = 6 - 1/2(6) = 3$ , so  $SP = (3,0,3)$
- e)  $SV = (0, 2.5, 0)$
- f) Primal:  $\text{Min } Z = \underline{C}X$       S.t.:  $\underline{A}X \geq \underline{B}, X_j \geq 0$   
Dual:  $\text{Max } Y = \underline{B}'\underline{W}$       S.t.:  $\underline{A}'\underline{W} \leq \underline{C}', W_j \geq 0$   
 $\text{Max } Y = 6W_1 + 2W_2 + 3W_3$   
Subject to  $W_1 + W_3 \leq 6$   
 $8W_2 + 3W_3 \leq 12$   
 $2W_1 + 3W_2 \leq 6$        $W_j \geq 0$
- g)  $Y^* = 27$
- h)  $W^* = (3,0,3)$
- i) Adding one unit of  $W_2$  would make the second constraint binding: to avoid this non-marginal complication, let us add  $1/10$  unit  $W_2$ .  $Y$  would increase  $2/10$ , but constraint #3 forces us to reduce  $W_1$  by  $3/20$ , which reduces  $Y$  by  $9/10$ . The reduction in  $W_1$  allows an increase of  $3/20$  in  $W_3$ , adding  $\Delta Y$  of  $9/20$ . Altogether, then  $\Delta Y = 5/20$ , and  $\Delta Y/\Delta W_2 = 5/2$ .  $OC = (0,2.5,0)$

j)  $SP_1 = 3$  (one more  $W_3$ ) ;  $SP_2 = 0$  (not binding);  
 $SP_3 = 3$  (1/2 unit more  $W_1$ ) - 1.5 (1/2 unit less than  $W_3$  in constraint #1) = 1.5 ;  $SP = (3, 0, 1.5)$

k)  $SV = (0, 3, 0)$

l) Primal variables (X's) = dual SP's  
primal SP's = dual variables (W)

At optimum:

Primal objective function (Z) = dual objective function (Y)

Primal OC's = dual SV's ; primal SV's = dual OC's

### 6.6 Iron Alloys

a)  $\$238-17 = \$221/\text{ton}$

b) The cost to you would be 0.10 (opportunity cost #1). At  $\$38/\text{ton}$ , scrap #1 has an opportunity cost of  $\$6.7/\text{ton}$ . If you purchased scrap #1 at  $\$38$ , you would lose only  $\$5.7/\text{ton}$ . Therefore:  $OC = 5.7$  and the deal would cost you 0.10 ( $\$5.7$ ) =  $57\text{¢}/\text{T}$ . The benefit of the deal would be 0.2 (SP on carbon) =  $48\text{¢}/\text{T}$ . You would lose  $9\text{¢}/\text{T}$  on the deal: you would be better off to sell your normal product at normal prices.

c) .2% more chromium will cut costs by .2 (6.1) =  $\$1.22/\text{T}$   
0.1% more of both chromium and manganese will save  $\$1.11/\text{T}$   
You cannot use more than .1% more manganese without overrunning your range, so .2% more manganese will not work (especially since you know that the SP decreases as the constraint is relaxed).

### 6.7 Gravel Pit

a) We will ship as little as possible.

b)  $X_{26}$  has an  $OC = \$.17/\text{T}$  ;  $.17/\text{T}(100\text{T}) = \$17$  more (assumes no other constraints become binding in the 100T change).

c) The shadow price on her demand is  $\$1.20/\text{T}$ . However, this is only true for  $b \geq 1602$ , and thus for  $2147 - 1602 = 545\text{T}$  the SP is  $\$1.20/\text{T}$ . Therefore, at least  $\$654$  is saved. Since the constraint is being relaxed, the SP must fall: perhaps to  $\$1.19$ , perhaps to zero. Therefore, refund would be:  $\$654 \leq R \leq 840$ .

d) Same situation as (c), except that constraint is tightening. The cost would be at least  $\$45$  and may be infinite.

### 6.8 Food Distribution

- a) Since link 1 to 6 is not operating at capacity, i.e., the constraint is not binding, X must be zero : no shadow price.
- b) The present cost on link 2 to 5 = 143 Pesos, with an opportunity cost of 21 Pesos. This means that this link would be used if its cost were reduced by 21 Pesos to 122 Pesos, or less. Since these improvements will reduce the cost to 120 P, this link would be used in the optimal distribution.
- c) The shadow price on link 1 to 5 = 12. Since an increase in capacity from 63 to 68 on this link is within the range of the specified SP, an increase (relaxing) of the constraint of 5 units is worth  $5(12) = 60$  P.
- d) Growing vegetables at town 6 would reduce demand there for veggies transported in from the countryside : the demand constraint would be relaxed. The shadow price on the demand at city 6 is 113 Pesos. One bushel of veggies at 100 Pesos would save the shipping charges of 113 Pesos for a net savings of 13 Pesos. But, the SP of 113 is valid only for demand  $\geq 38$ , and the production of four bushels of veggies would bring demand for shipped veggies down to 36 bushels. Since the SP of the two bushels below 38 (37th Z 38th) will be less than 113 and may be as low as zero, we cannot determine if or not growing bushels pays. We do know that the cost of shipping these four bushels is between 226P (if SP falls to zero) and 450 P (if SP falls to 112), but unless we rerun the program at a reduced demand of 36 bushels we simply cannot say if growing four bushels at home pays financially.

### 6.9 Gelderland

- a) The shadow price for the actual cost constraint is 15G/\$. Thus the nation can afford to spend  $(15G/\$)(\$1M) = 15MG$ .
- b) Relaxing the standards by 1% (i.e. to  $\leq 2\%$ ) is tantamount to taking a 1% chance of a 10 M.Gelders cost. Therefore, the choice here is between a sure cost of 600,000 Gelders and a cost of 10 M.Gelder with probability 0.01 (EV = 100,000 Gelders). Taking the risk seems reasonable.
- c) There are no financial benefits from finishing early (0 G/year), so why pay more money for labor?
- d) There is not enough information to solve the problem. Need information about what kind of ships and if they can be served by barges. How long do barges last? Do probabilities of spills increase?

