Exercise 4.3

Question

4.3. *Marginal Analysis III*

Given the production function: $Z = 10X^{0.3}Y^{0.4}$

and the input cost function: $C = 3X + 2Y^2$

(*a*) Find the marginal products and costs.

(*b*) State the optimality criteria for this situation.

(*c*) Write an expression in X and Y for the marginal rate of substitution of X for Y.

(*d*) Find the expansion path.

(*e*) Find the cost-effectiveness function.

Solution from Manual

4.3 Marginal Analysis III

a) $MP_X = 3X^{-0.7}Y^{0.4} = .3Z/X$ $MP_Y = 4X^{0.3}Y^{-0.6} = .4Z/Y$

$MC_X = 3$ $MC_Y = 4Y$

b) At optimum: $MP_X/MC_X = MP_Y/MC_Y$

c) $MRS_{XY} = \Delta X/\Delta Y = -MP_Y/MP_X = -(.4Z/Y)/(.3Z/X) = -4X/3Y$

d) At optimum: $(.3Z/X)/3 = (.4Z/Y)/4Y$, or $Z/X = Z/Y^2$

So, $X^* = (Y^*)^2$.

e) We wish to find $C = f(Z)$ along the optimum path $X^* = (Y^*)^2$

substituting $X = Y^2$ into $C(X,Y)$ and $Z(X,Y)$, we get $C = Z^2/20$.

Additional Notes

a) and c)

$$MP_x = \frac{\partial Z}{\partial X} = \frac{0.3Z}{X} \quad MP_y = \frac{\partial Z}{\partial Y} = \frac{0.4Z}{Y}$$

$$MC_x = \frac{\partial C}{\partial X} = 3 \quad MC_y = \frac{\partial C}{\partial Y} = 4Y$$

$$MRS = \frac{\Delta X}{\Delta Y} = -\frac{MP_y}{MP_x} = -\frac{4X}{3Y}$$

b) and d) The optimality condition can be stated as:

$$\frac{MP_x}{MP_y} = \frac{MC_x}{MC_y} \Rightarrow \frac{3Y}{4X} = \frac{3}{4Y}$$
$$\Rightarrow 12Y^2 = 12X \Rightarrow Y = X^{0.5}$$

And so $Y = X^{0.5}$ describes the expansion path.

e) We need to find the cost function (beware: different from input cost function!):

$$\textit{We know } Y = X^{0.5}$$
$$\textit{then } Z = 10X^{0.3}\left(X^{0.5}\right)^{0.4} = 10X^{0.3}\left(X^{0.2}\right) = 10X^{0.5}$$
$$\Rightarrow X = \left(\frac{Z}{10}\right)^2$$
$$\textit{Now,}$$
$$C = 3X + 2Y^2 = 3X + 2\left(X^{0.5}\right)^2 = 3X + 2X = 5X$$
$$\Rightarrow C = 5\left(\frac{Z}{10}\right)^2 = \frac{Z^2}{20}$$