Exercise 4.13

Question

4.13. *Efficient Design*
Given the production function: $Z = 4^{0.7}X^{0.3}Y^{0.5}$
and the input cost function: $C = 38 + 3X + 10Y^2$
(*a*) Does the production function define a convex feasible region? Explain.
(*b*) What type of returns to scale does the production function exhibit?
(*c*) Define and calculate the marginal rate of substitution.
(*d*) Determine the expansion path.
(*e*) Given a budget of C = \$60, what is the maximum output that can be achieved What combination of inputs is required to achieve this output at C = \$60?

Solution from Manual

4.13 Efficient Design

a) Since all a_i are less than one, the MP's are decreasing. The RTS are also decreasing because $\Sigma a_i < 1$. Therefore, the feasible region is convex.

b) $\Sigma a_i = .8$ Decreasing RTS.

c) The MRS is the rate at which marginal increases in an input must be substituted for marginal decreases in another input so output remains constant:
$MRS_{XY} = -(.5/Y)/(.3/X) = -5X/3Y$

d) $MP_X/MC_X = MP_Y/MC_Y$ $[(.3/X)Q]/3 = [(.5/Y)Q]/20Y$
$20Y^2 = 5X$; $Y = 0.5X^{0.5}$

e) The cost-effectiveness function is: $C = 38 + 22(Z/4)^{(2/1.1)}$
Setting $C = 60$, $Z = 4$
Since $Z = 4Y^{1.1}$ and $X = 4Y^2$: $Y = 1$, $X = 4$

Additional Notes

a) The following graphs are generated with the spreadsheet ProdFun + MargAnal.xls:

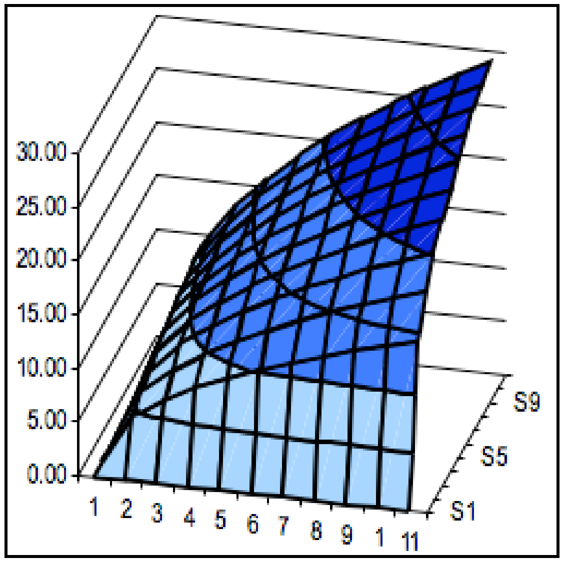


Figure 1: 3D view of the production function

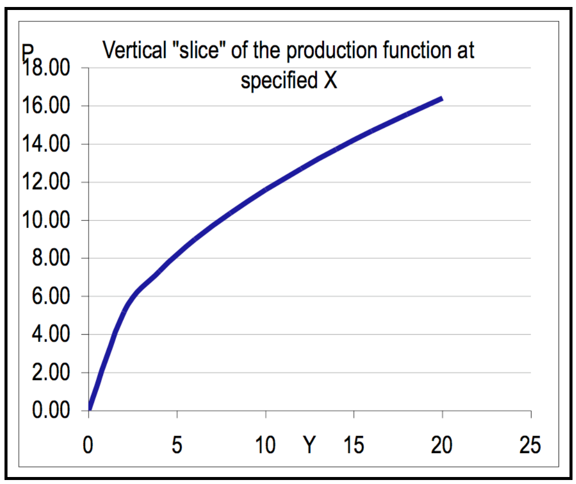


Figure 2: Vertical slice of the production function along Y axis at specific X

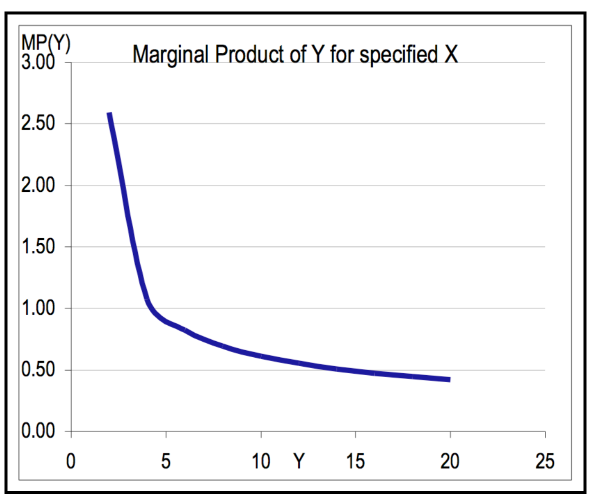
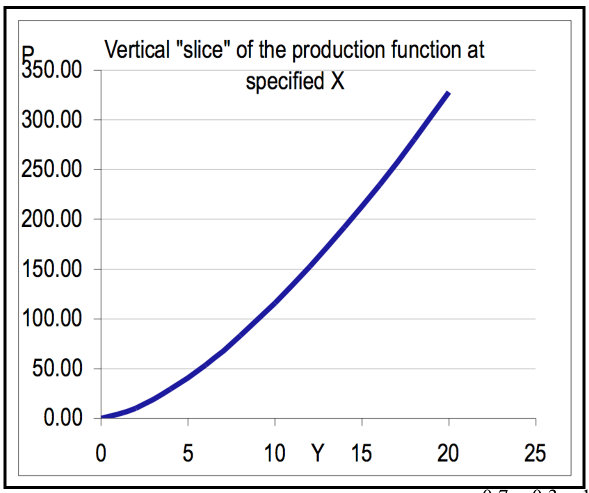


Figure 3: Marginal product of Y for specified X showing decreasing MP

Two distinct properties of production functions can be looked at to determine whether the function exhibits a convex feasible region. First, if marginal products are decreasing for all inputs, this means the production function exhibits a shape similar to that shown in Figure 2. This is because as inputs increase, it is harder to produce additional increments of output, which is typical of convex production functions. When differentiating marginal products, if exponents are less than 1, this means exponents in the equation for marginal products will be negative, thus showing a behavior as the one shown in Figure 3.

Second, decreasing returns to scale ensures that the production function will not exhibit non-convex behavior for any input value. Increasing returns to scale typically characterizes a curve with exponential-like behavior, so that as inputs double, output more than doubles (see figure below). The returns to scale can be assessed by looking at the exponents of the production function: if the sum is higher than 1, there is increasing RTS, if equal to 1 there is constant RTS, and if lower than there is decreasing RTS.



Production function for a specified X of function $Z = 4^{0.7}X^{0.3}Y^{1.5}$, which exhibits increasing RTS

If these two conditions are satisfied, then the feasible region of the production function must be convex. Therefore, marginal analysis can be applied.

d) The expansion path is the locus of all optimum designs for every level of output Y. At optimality, the slope of the isoquant (as found using the marginal product) should equal the slope of the budget line (as defined by the marginal cost). Therefore:

$$MP_x = \frac{\partial Z}{\partial X} = \frac{0.3Z}{X} \quad MP_y = \frac{\partial Z}{\partial Y} = \frac{0.5Z}{Y}$$

$$MC_x = \frac{\partial C}{\partial X} = 3 \quad MC_y = \frac{\partial C}{\partial Y} = 20Y$$

$$MRS = \frac{\Delta X}{\Delta Y} = -\frac{MP_y}{MP_x} = -\frac{5X}{3Y}$$

Now, the optimality condition can be stated as:

$$\frac{MP_x}{MP_y} = \frac{MC_x}{MC_y} \Rightarrow \frac{3Y}{5X} = \frac{3}{20Y}$$

$$\Rightarrow 60Y^2 = 15X \Rightarrow Y = \frac{X^{0.5}}{2}$$

And so Y = $X^{0.5}/2$ describes the expansion path.

e) We need to find the cost function (beware: different from input cost function!):

$$We \;\; know \;\; Y = \frac{X^{0.5}}{2}$$

$$then \;\; Z = 4^{0.7}X^{0.3}\left(\frac{X^{0.5}}{2}\right)^{0.5} = 4^{0.7}X^{0.3}\left(\frac{X^{0.25}}{2^{0.5}}\right) = 2^{0.9}X^{0.55}$$

$$\Rightarrow X = \left(\frac{Z}{2^{0.9}}\right)^{\frac{1}{0.55}}$$

Now,

$$C = 38 + 3X + 10\left(\frac{X^{0.5}}{2}\right)^2 = 38 + 3X + \frac{10X}{4} = 38 + \frac{22X}{4}$$

$$\Rightarrow C = 38 + \frac{22}{4}\left(\frac{Z}{2^{0.9}}\right)^{\frac{1}{0.55}}$$

If C = \$60, then Z = 4 by completing the algebra.

As for the combination of inputs required to get C = \$60,

$$Since\ \ Z = 2^{0.9} X^{0.55},\ \ then\ \ if\ \ \ Z = 4 \Rightarrow X = 4$$

$$Since\ \ Y = \frac{X^{0.5}}{2} \Rightarrow Y = 1$$

Thus we need a design with X = 4 and Y = 1 to achieve the most efficient design, the one producing the highest output (Z = 4) given a budget of $60.