Option Valuation (Lattice)

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Outline

- Recall: Types of Options Mantra
- Lattice Method of Representing Uncertainty
  - Formulation and Key attributes
- Application to Decision Analysis
- Application to Financial Options
  - Portfolio Replication interpreted as "probabilities"
  - Arbitrage possibility enforces risk free rate
  - Analysis with R_f and "risk neutral probabilities"

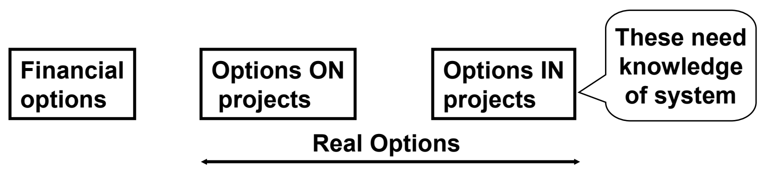
Two Types of Options

- Financial Options
  - These concern contracts on traded assets (such as stock, bonds, foreign exchange, bonds, etc.)
  - These are most common
  - Largest, oldest (to 1970s) and most sophisticated literature
- "Real" Options
  - These concern projects which may or may not produce a traded asset (Ex: a copper mine)
  - Least talked about
  - Recent literature (most from 1990's)

Two Types of Real Options

- Options "ON" projects
  - These do not concern themselves with system design
  - They examine options to open, close, delay a project
  - EX: the option to open a mine (Antamina case)
  - Most <u>common in literature</u>
- Options "IN" projects
  - These involve changing the technology some way
  - These require detailed understanding of system
  - EX: design elements that permit changing altitude (and capacity) of communication satellites
  - Most <u>interesting to system designers</u>

Total: Three Types of Options



- These distinctions effect the valuation method
- The question is: which approaches best in each circumstance?

Valuing Options: Concepts

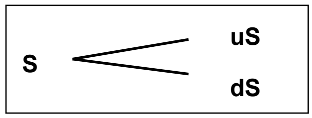
- Financial Options:
  - ONLY Correct Way: “Arbitrage Enforced Pricing”
    - * Ability to construct replicating portfolio
    - * ➔this portfolio defines price that must prevail
    - * If not, others in market can make you a loser
- Real Options:
  - Multiple approaches (see Borison), for example:
    - * Based on replicating portfolio if data good (Merck)
    - * Decision analysis if no historical data (Kodak)
    - * Simulation using either or both (Antamina)
  - Lattice analysis a key tool in overall context

Lattice Method (Binomial Tree)

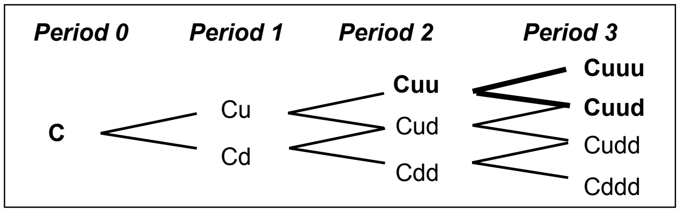
- Reproduces uncertainty over time to simulate actual sequence of possibilities
- Approximates price changes as sequence of increases and decreases over stages
- Accuracy depends on number of stages – can be very detailed and accurate
- Has special features that permit easy solution using Dynamic Programming

Lattice Construction: 1 Stage module

- Lattice is a sequence of single stage modules
- Each shows changes in state of system
- Binomial if only 2 possibilities: up or down changes with probability, P_u and P_d
- State of system correspondingly changes by a multiplicative factor up or down, u or d



Lattice: Many stages (Decision Tree)



- **Lattice is sequence of single-stage modules**
  - **- States coincide (“up then down” path gives same state as “down then up”)**
  - **- Number of states increases linearly (1,2, 3, 4….) , not exponentially (1, 2, 4, 8…)**
  - **- System state defines Node – PATH INDEPENDENT**

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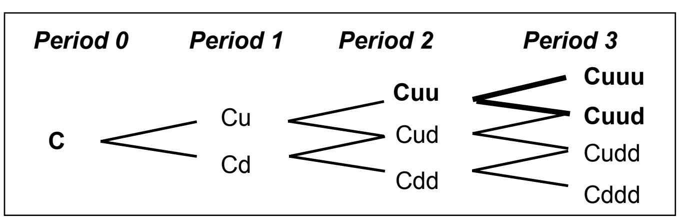
Implications of Path Independency



- **P.I. permits implicit enumeration of paths**
  - **- You do not have to examine all paths to a node**
  - **- E.g.: Best decision at Cud defined only by state of system, not by paths to Cud. When you have found it, you have implicitly considered paths to get there**
  - **- Analysis linear in # of stages (not exponential)**

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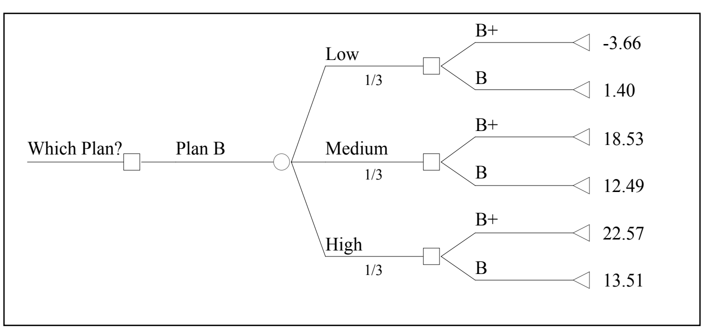
Analysis of Lattice

- Lattice analysis is as for decision tree:
  - Start at end (right-hand side)
  - Determine best choice at that stage
  - Roll back to previous stage, and repeat
- When P.I. holds, a special efficient method is possible: Dynamic Programming (see text)
  - Defines Recursion formula that expresses one stage as function of next (Bellman Equation)
  - Automates process
- Can be done with Excel add-ins

Application to Decision Analysis

From Homework: Optimal Plant Investment (2)



Why should analysis be limited to a decision in year 3?
To only high, medium and low demand scenarios?

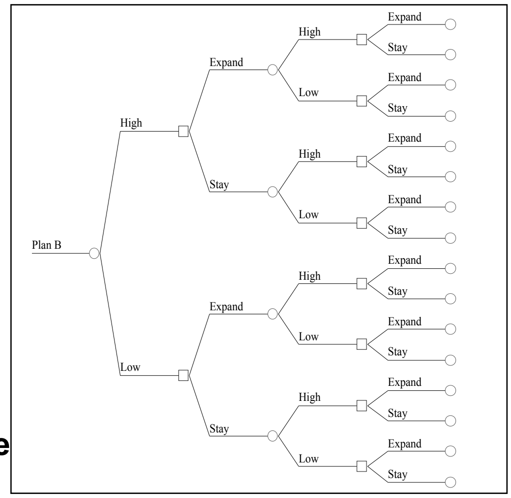
More detail with Decision Analysis

Can be done

- **Make Δt smaller**
- **Over Δt, project demand up or down**
- **Over entire horizon, demand can cover a known distribution**

But

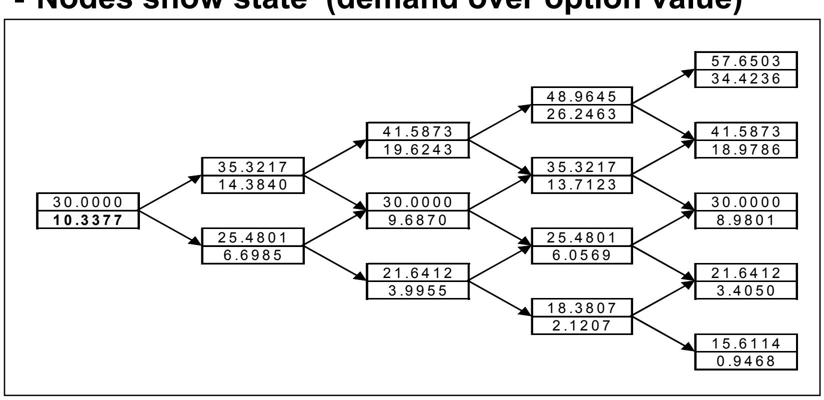
- **Tree ➔ messy bush**
- **Even for simple case**



Solution by Lattice Analysis

Analyze lattice, get results as shown below

- **Nodes show state (demand over option value)**



Application to Financial Options

- The correct analysis of financial options requires arbitrage enforced pricing
  - Previously used example demonstrates point
- Thus, traditional probabilities (defined by logic, frequency or estimates) NOT appropriate
- How then does lattice analysis work?
- Answer is TRICKY! Sets up portfolio to look like probabilities, but the are not really!

Case from previous lecture

- Idea: determine Fair Value of Option, C

-- If end of period asset price S > K, strike price: payoff of the option = S - K

-- If end of period asset price S < K, strike price: payoff of the option = 0

	Start	End	End
Asset Price	100	80	125
Buy Call Strike = 110	- C	0	(125 - 110) = 15

Table of Portfolio Cost and Payoffs

	Start	End	End
Asset price	100	80	125
Buy Stock	-100	80	125
Borrow Money	80/(1+r)	- 80	- 80
Net	-100 + 80/(1+r)	0	45

The Portfolio of Buying Asset with Loan replicates option

Comparing Costs and Payoffs of Option and Replicating Portfolio

- If S < K, both payoffs = 0 and are equal
- If S > K, portfolio payoff is a multiple of call payoff. In this case, ratio is 3:1
- Thus, payoff of 3 calls = portfolio payoff
- Note: arbitrage is riskless so r = risk-free rate

Period	Start	End	End
Asset Price	100	80	125
Buy Call	- C	0	(125-110) = 15
Buy Asset And Borrow	-100 + 80/(1 + r)	0	45

Value of Option

- Value of Option = Value of Portfolio
- This is easy to define, using risk-free rate, Rf
  - Calculation below assumes Rf = 10% (for easy calculation)
- C = (1/3)[-100 + 80/ (1 + Rf)] = $ 9.09

Period	Start	End	End
Asset Price	100	80	125
Buy 3 Calls	- 3C	0	45
Buy Asset And Borrow	-100 + 80/(1 + r)	0	45

Interpreting Example as 1-stage lattice

- Value of Option is:
  - At Start: C this value is to be found
  - At End: either Cu = 15 or Cd = 0 (uncertain outcomes)
- Likewise, value of Asset is:
  - At Start: S
  - At End: either uS or dS
- To find C, we have to find share of asset ("x") and loan ("y") in replicating portfolio
- We solve: xuS + yR = Cu and xdS + yR = Cd

=> x = (Cu - Cd) / S(u - d)
=> y = (1/R) [uCd - dCu] / (u - d)

Solving Lattice for Value of Option

Portfolio Value = Option Price

= [(R - d)Cu + (u - R)Cd] / R(u-d)

= [(1.1- 0.8) (15) + (1.25 - 1.1)(0)] / 1.1(1.25 - 0.8)

= [0.3(15)] / 1.1(.45) = 10 / 1.1

= 9.09 as before

Rewrite formula, using factor "q" = (R - d) / (u - d)

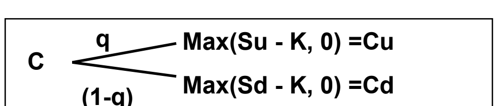
Option Price = [(R - d)Cu + (u - R)Cd] / R(u-d)

= (1/R) [qCu + (1-q) Cd)]

Note that this looks just like a probabilistic binomial stage with probability "q" and "1-q" !!

Arbitrage seen as Expected Value

- **Extraordinary interpretation!**
- **Option value = "expected value" over "risk-neutral probabilities" q and (1 - q)**
- **Yet "q" is defined in terms of spread**
- **Actual probabilities do not enter calculation!**
- **Graphically:**



General Multi-Stage Procedure

- Lattice analysis using q and (1- q)
- At each node, compare:
  - Option value
  - Payoff of Immediate exercise
- … and Determine Optimal decision
  - Hold option for another period
  - Exercise immediately
- Issue: values of "u" and "d" , that specify "q"

Determining "u", "d": Assumptions

- These parameters reflect range of possible outcomes, thus an assumed pdf
- Usual assumption is that pdf is random, because
  - Project risks can be avoided by diversification
  - Thus only looks at market risk
  - Assumes efficient markets thus no bias
  - Thus error is random or "white noise"
- Accepts that
  - growth trends may exist
  - Values do no go negative
- Thus, usual assumption is that random variations is
  - log- normal, with standard deviation σ
  - Wiener process or Generalized Brownian Motion

Determining "u" ,"d": Formulas

- With Usual assumption that random variations are log- normal scale, with standard deviation σ

$$u = e \exp (\sigma \sqrt{\Delta t})$$

$$d = e \exp (- \sigma \sqrt{\Delta t})$$

Where Δt is the fraction of a year (e.g, 1 month = 1/12) , so that

$$R = 1 + Rf * \Delta t$$

Summary for Application to Financial Options

- Binomial model is a recursive technique
  - Starts with end-period values, works back to present
  - Tedious, but usually automated
- Note similarity to NPV
  - Estimate cash-flows (end-of-period option value)
  - Discount to present (using risk-free rate)
- But Model is very different from NPV analysis!
  - Payoffs are created by the factors "u" and "d"
  - The "probability" "q" is not an actual probability; it is derived from Arbitrage-enforced pricing
  - Discount rate is "risk free" – due to Arbitrage

Summary

- Lattice Method similar to a Decision Tree,
- … but with specific structure
  - Nodes coincide
  - Values at nodes defined by State of System
  - Thus “path independent” values
  - Enabling rapid analysis (Dynamic Programming)

- Lattice Analysis widely applicable
  - With actual probability distributions
  - For Financial options using factors that are
    - called “risk-neutral probabilities”
    - But actually represent relative share of loan and stock in replicating portfolio