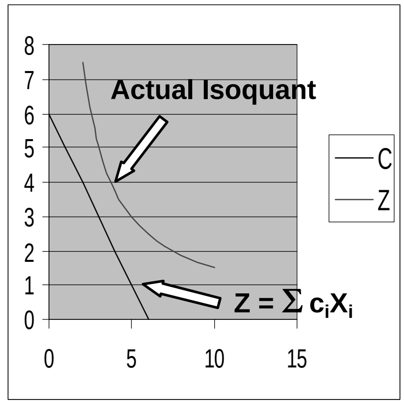
Linear Programming
in Practice

Linear Programming in Practice

- Essential Issue: To model non-linear reality with linear equations
  - –Activities
  - –Piece-wise linear approximations
  - –Fixed charges
- An issue that may come up: Duality
- Some Example applications

Activities (1)

- **Motivation:**
- **If we use a standard production function**
- **$f(\underline{X}) = \Sigma c_iX_i = Z$**
- **resources ⇨ output**
- **We are not able to represent typical production function with diminishing marginal returns and non-linear isoquants**



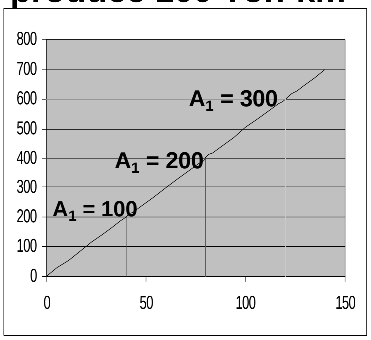
Activities (2)

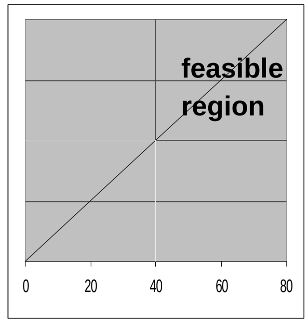
- **Concept**
  - –**An activity is a**
  - –*Specific way to use resources in fixed proportions*
- **Physical interpretation is direct, e.g.:**
  - –**an aircraft using pilots, fuel / ton-km**
  - –**a machine requiring labor, materials per unit product**
- **Think of activities as intermediates between resources and output**

resources ⇨ activities ⇨ output

Example for 1 Activity

transport process A_1 uses 40 persons, 200 gallons to produce 100 Ton-km





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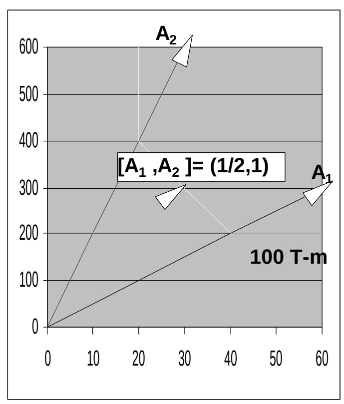
Two Activities

A_1 = (40p,200g) ==> 100 T-m

A_2 = (10p,200g) ==> 50 T-m

A_1 = ½ (20, 100) => 50
A_2 = 1 (10, 200) => 50
[A_1, A_2] = [½, 1]
(30,300) => 100
Note: Isoquant horizontal and vertical above, below activities -- Why?



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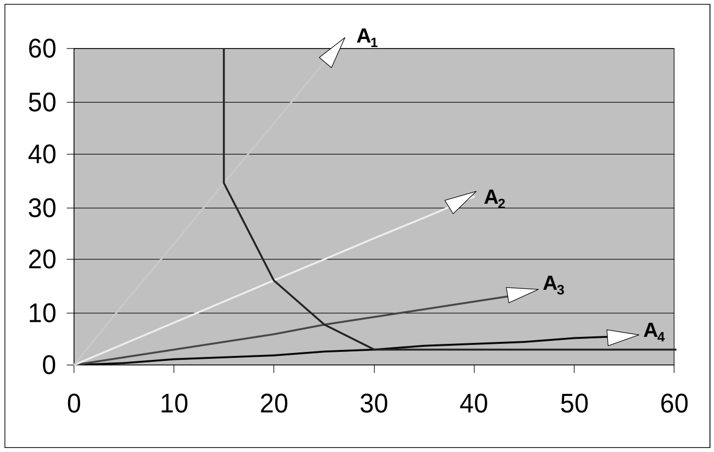
Many Activities



LP Formulation with Activities (1)

- **Example: Maximize Profits from Production of Alloys,**
  - –**3 possible processes**
  - –**limited by resources on hand (Chrome and Carbon)**
  - –**Different profitability for each process**
- **Optimize: Profit =** $\Sigma c_i P_i$ **-- subject to constraints**

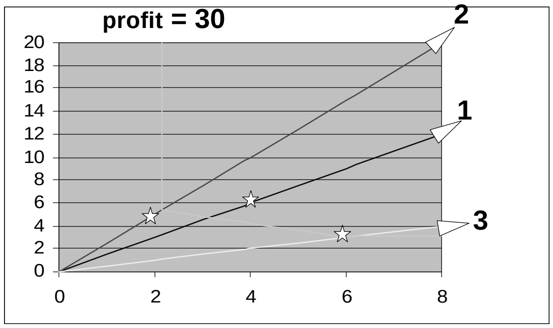
Element	Process 1	Process 2	Process 3
Cr	6	5	3
C	4	2	6
Profit	30	28	29

LP Formulation with Activities (2)

$$\max Z = 30\,P_1 + 28\,P_2 + 29\,P_3$$

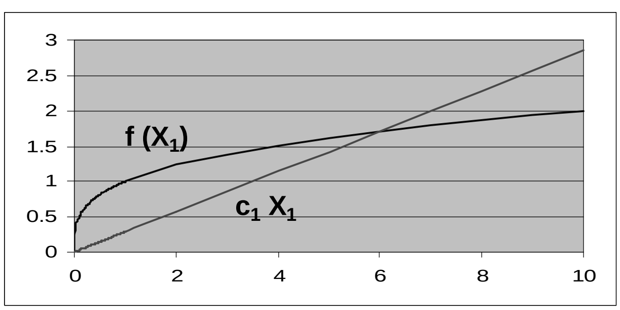
$$\text{s.t.} \quad 6\,P_1 + 5\,P_2 + 3\,P_3 \leq 26 \ \text{(Cr)}$$

$$4\,P_1 + 2\,P_2 + 6\,P_3 \leq 7 \ \text{(C)}$$



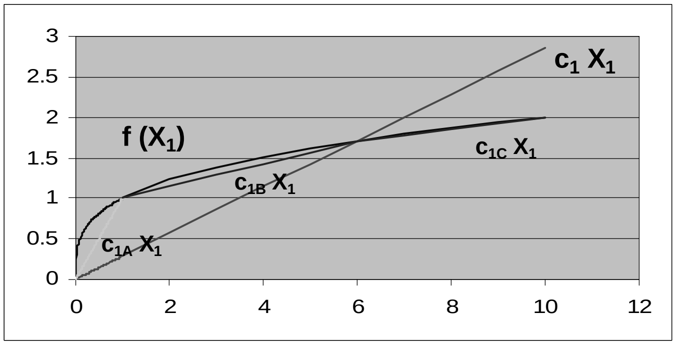
Piece-Wise Linear Approximations (1)

- **Motivation:**
  - –Returns to scale generally non-linear
  - –Straight line approximations are inaccurate



Piece-Wise Linear Approximations (2)

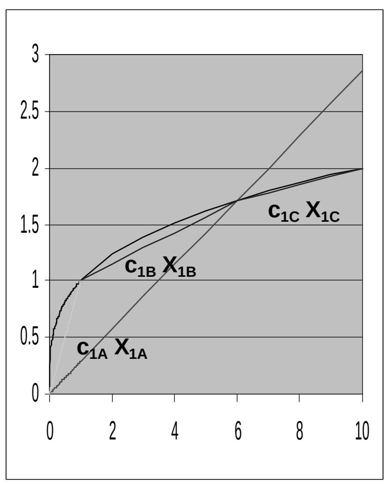
- Concept:
  - Represent $f(X_1)$ with several lines



Piece-Wise Linear Approximations (3)

Implementation Notes:

- X1 must be redefined as several variables: X_{1A}, X_{1B}, ...
- These new variables must not overlap, so $X_{1A} < X_{1B}$, etc.
- New variables and constraints make the LP larger and, thus more expensive



Piece-wise Linear Approximations (4)

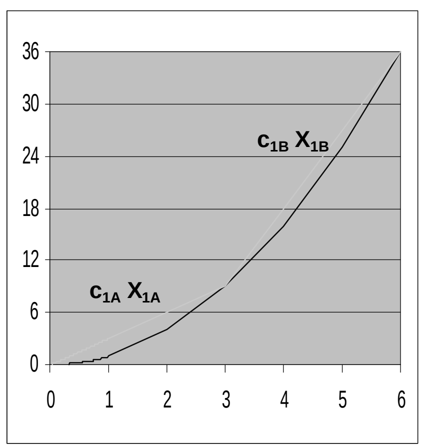
- Given: Max $Z = f(X_1) + 4X_2$
  s.t. $3X_1 + 6X_2 \le 8$
- Piece-wise linear approximation gives:
  - $X_1 \Rightarrow X_{1A} + X_{1B}$
  - X_{1A}, X_{1B} have same a_{ij} as X_1
  - $c_1 = c_{1A}$, c_{2A}
  - X_{1A} < cutoff X value between X_{1A} and X_{1B}, X'
- Thus: Max $Z = c_{1A} X_{1A} + c_{1B} X_{1B} + 4X_2$
  s.t. $3 X_{1A} + 3 X_{1B} + 6X_2 \le 8$
  $X_{1A} \le X'$

Piece-wise Linear Approximations (5)

- Key Limitation:
  - ONLY works for convex feasible region!
- Why?
  - What if $c_{1B} > c_{1A}$? (see fig)

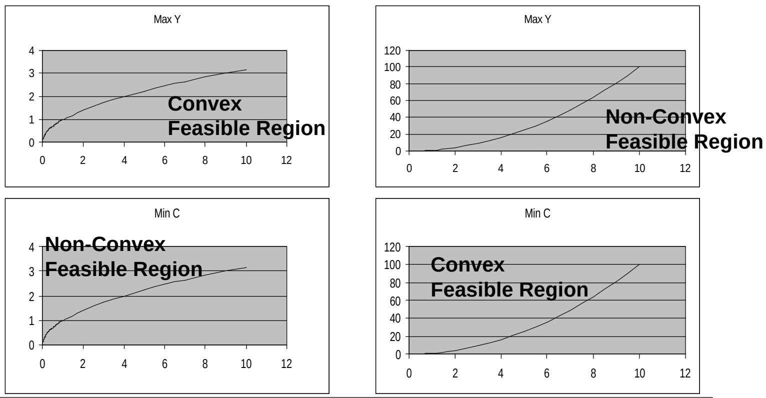
Max $Z = c_{1A} X_{1A} + c_{1B} X_{1B} + 4X_2$

- The LP will select X_{1B} before X_{1A}
Result may be meaningless!



Convex Feasible Regions Review:

Piecewise linear approximation works when FR is convex



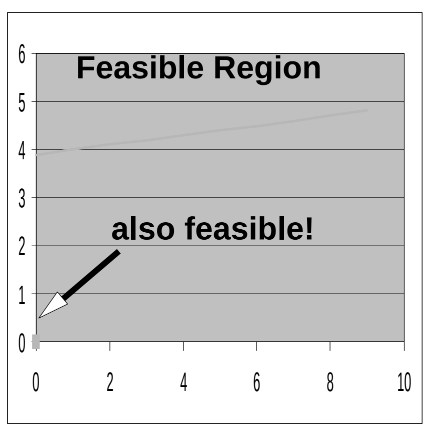
Fixed Charges

- **Example: Warehousing**
  - **Cost = fixed rent, etc. + variable**
  - **__Unless__ you choose not to operate it!**

$f(X_1) = c_0 + c_1X_1 \quad X_1 \geq 0$

$f(X_1) = 0 \quad X_1 = 0$

- **LP generally cannot handle fixed charges**
  - **__Exception:__**
  - **All $X_i > 0$; $X_i \neq 0$**
  - **then subtract Σc_0**
  - **and optimize**



Duality

- **Concept:**
  - **A "dual" is a mirror-image form to another problem (the "primal")**
  - **If primal = max; then dual = min**
    If primal = min; then dual = max
  - **Dual contains all information of the primal, but in a different format**
  - **Optimum value of primal = optimum value of dual**
- **Example:**
  - **Primal: maximize output subject to budget limitations**
  - **Dual: minimize costs subject to output requirements**

LP Duality

- **Mathematics:**
  - **Given a Primal:**

Optimize:	$Z = \underline{c}\ \underline{X}$
subject to:	$\underline{A}\ \underline{X} \leq \geq \underline{B}$

  - **Dual is:**

Optimize:	$Y = \underline{B}^T\underline{W}$
subject to:	$\underline{A}^T\underline{W} \leq \geq \underline{c}^T$

- **Change of dimensionality between primal & dual:**
  - $\underline{c}^T$ **and** $\underline{B}$ **have different number of variables**
- **Can use duality to:**
  - **Reduce size of constraint matrix**
  - **Speed up LP solution**

LP Duality - Example (1)

Primal: Max: $Z = X_1 + 2X_2 + 3X_3$

s.t. $4X_1 + 2X_2 \leq 5$

$6X_1 + 7X_2 + 9X_3 \leq 12$

$A = \begin{matrix} 4 & 2 & 0 \\ 6 & 7 & 9 \end{matrix}$ $\quad A^T = \begin{matrix} 4 & 6 \\ 2 & 7 \\ 0 & 9 \end{matrix}$ $\quad B = \begin{matrix} 5 \\ 12 \end{matrix}$ $\quad B^T = [5\ 12]$

$C = [1\ 2\ 3]$ $\quad C^T = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$

So: Max: $Z = C X$

s.t. $A X \leq = \geq B$

LP Duality - Example (2)

Primal: Max: $Z = X_1 + 2X_2 + 3X_3$

s.t. $4X_1 + 2X_2 \leq 5$

$6X_1 + 7X_2 + 9X_3 \leq 12$

Dual: Min: $Y = 5W_1 + 12\,W_2$

s.t. $4W_1 + 6W_2 \geq 1$

$2W_1 + 7W_2 \geq 2$

$9W_2 \geq 3$

So: Max: $Z = C X$ $\quad \Rightarrow \quad$ Min $Y = B^T W$

s.t. $A X \leq = \geq B$ $\quad$ s.t. $A^T \leq = \geq C^T$

LP Duality - Interpretation of Results

–**Primal**:

Max: $Z = 3X_1 + X_2 + 8X_3$

s.t. $X_1 + X_3 \leq 4$

$X_1 + X_2 + X_3 \leq 7$

$2X_2 + X_3 \leq 8$

–**Dual:**

Min: $Y = 4W_1 + 7W_2 + 8W_3$

s.t. $W_1 + W_2 \geq 3$

$W_2 + 2W_3 \geq 1$

$W_1 + W_2 + W_3 \geq 8$

$X^* = \{0,2,4\}$
$SP^* = \{7.5,0,0.5\}$
$OC^* = \{4.5,0,0\}$
$SV^* = \{0,1,0\}$
$Z^* = 34$

$W^* = \{7.5,0,0.5\}$
$dSV^* = \{4.5,0,0\}$
$dSP^* = \{0,2,4\}$
$dOC^* = \{0,1,0\}$
$Y^* = 34$

Primal-Dual Relationships in Solution

Primal		Dual
Decision Variables	→	Shadow Prices
Shadow Prices	→	Decision Variables
Opportunity Costs	→	Slack Variables
Slack Variables	→	Opportunity Costs

Some Real-World Applications (1)

- Airline Scheduling
  - Objective: Minimize Cost
  - Constraints:
    - Number of Aircraft
    - Available time on Aircraft before Maintenance
    - Crews Available
    - Limits on Crew time on duty
    - Location of crews and aircraft
    - Traffic between points, etc, etc

Barnhart, C., Johnson, E., Nemhauser, G., and Vance, P., (1999) "Crew Scheduling," Handbook of Transportation Science, Randolph W. Hall (editor), Kluwer Academic Publishers, Norwell, MA, pp. 493-521.

Some Real-World Applications (2)

- Production and Logistics
  - Objective: Minimize Cost / Maximize Throughput
  - Constraints:
    - Number and Capacity of Facilities
    - Connectivity of Network
    - Personnel restrictions
    - Location of crews and aircraft
    - Times orders are made
    - Delays in system, etc, etc,

Simchi-Levi, D., P. Kaminsky and E. Simchi-Levi, (1999) Designing and Managing the Supply Chain: Concepts, Strategies and Case Studies, Irwin/McGraw-Hill. 2000 Book-of-the-year and Outstanding Publication awards by the Institute of Industrial Engineering

Summary on Applications

- Special Formulations required so that LP can represent realistic problems
  - Activity representations
  - Piece-wise linear approximations
  - Integers (not discussed here)
- Much sophistication in mathematics possible
  - Duality gives a flavor
  - See Example applications
- However, LP basically deals with system models with known parameters, without risk

MORE WILL BE REQUIRED!!!