Introduction to Linear Programming (LP)

- Mathematical Programming Concept
- LP Concept
- Standard Form
- Assumptions
- Consequences of Assumptions
- Solution Approach
- Solution Methods
- Typical Formulations

Mathematical Programming (MP) Concept

- Definition
  - MP includes a range of powerful computer-based optimization methods
- Approach to Optimization
  - MP methods exploit peculiar features of the structure of a problem to get solutions efficiently
  - Different MP methods exploit different structures or features
  - Understanding which features apply to a particular problem -- and thus which method can apply -- is important

MP: Two important Categories

- Those Valid for Convex Feasible Regions
  - –Linear Programming, etc
  - –These use a local search routine to reach global optimum
  - –“Keep going up/down until reach top/bottom”
  - –Very efficient -- but not valid for non-convex regions
- Those Valid for Non-Convex Feasible Regions
  - –Dynamic programming, etc
  - –These "enumerate" solutions to discover optimum
  - –They “prune,” that is, eliminate, possible solutions because these can be shown to be “dominated”
  - –Computations limit applicability to special situations, but good for options analysis

LP Overview

- Special form of mathematical programming
  - –Equations must be linear
- Uses simple solution procedures
  - –Linear algebra
- Very powerful
  - –Extremely large problems
  - –100,000 variables
  - –1000's of constraints
  - Note difference!
- Useful design information by Sensitivity Analysis
  - –Answers to "what if" questions

Standard Form of LP - Three Parts

- **Objective function (OF)**
  - **maximize or minimize**

⇒

- **$Y = \Sigma c_i x_i$**
- **$Y = c_1x_1 + c_2x_2 + ... + c_nx_n$**

X_i known as decision variables

- **Constraints**
  - **subject to:**

⇒

- **$a_{11}x_1 + a_{12}x_2 + ... < = > b_1$**
- **$a_{21}x_1 + a_{22}x_2 + ... < = > b_2$**
- **$a_{31}x_1 + a_{32}x_2 + ... < = > b_3$**

- **Non-Negativity** ⇒ **$x_i \geq 0$ for all i**

Standard Form of LP -- Summary

Optimize $Y = \underline{c}\ \underline{X}$

subject to: $\underline{A}\ \underline{X}$ (< or = or >) $\underline{b}$

$\underline{X} \geq 0$

Three LP Assumptions (1)

•**Linearity ; Additivity ; Non-Negativity**

▪**Linearity of Objective Function and Constraints**

–**Essential Condition is:**

$$f(k\underline{X}) = k\, f(\underline{X})$$

for example: $f(X) = 3 + 4X_1 + 2X_2$

is NOT linear in the LP sense

▪**Implies**

–**Constant returns to scale (only first order terms)**

–**No "fixed charges" (no constants)**

Three LP Assumptions (2)

▪**Additivity:**

$$f(X_1, X_2, \ldots, X_n) = f(X_1) + f(X_2) + \ldots + f(X_n)$$

–**no interactive effects among X_i terms**

–**assumes that individual segments of the problem operate as well independently as together**

▪**Non-Negativity**

$X_i > 0$

–**no fundamental difficulties except in particular situations**

Consequences of Assumptions

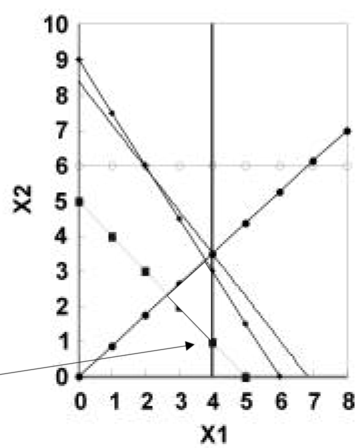
- Convexity of feasible region (if it exists!)
- Convex feasible region, with linear OF, implies:
  - Optimum will be on an edge of the feasible region
- Since edges are also linear
  - Optimum is at a corner point
  - (can be several in special cases)
- Corner points are a small, finite set, defined by solution of linear equations

<u>Bottom Line:</u> Assumptions imply the existence of an efficient solution strategy

Example for Assumptions

minimize: $Z = 3X_1 + 5X_2$

s.t.
$$X_1 + X_2 \geq 5$$
$$3X_1 + 2X_2 \leq 18$$
$$6X_1 + 5X_2 \leq 42$$
$$-7X_1 + 8X_2 \leq 0$$
$$0 \leq X_1 \leq 4$$
$$0 \leq X_2 \leq 6$$



Solution:
$X_1^* = 4$ $X_2^* = 1$

Example Solution from Excel Solver

	X(SUB1)		X(SUB2)	OBJ FCN
	4		1	17
EXPRESSION	CONSTRAINT		LIMIT	
X1 +X2	5	>=	5	
3 X1 + 2 X2	14	<=	18	
6 X1 +5 X2	29	<=	42	
(-7 X1) + 8 X2	-20	<=	0	
X1	4	>=	0	
X1	4	<=	4	
X2	1	>=	0	
X2	1	<=	6	

General Solution Approach

- Find a corner point
  - An "initial feasible solution"
- Proceed to improved corner points
- Stop when no further improvements are possible
- For large problems, a variety of more sophisticated approaches are used!

Solution Calculations

- To find a corner point
  - it is necessary to solve system of constraint equations
  - from linear algebra, this requires working with matrix of constraint equations, specifically, manipulating the "determinants"
  - Amount of effort set by number of constraints
- So number of constraints defines amount of effort
- This is why LP can handle many more decision variables than constraints

Solution Methods

- Simplex -- The 'textbook' method
  - For step 2, select improved corners
    - *Always goes to best corner*
    - *Searches until no further improvement possible*
  - Inefficient for real problems
  - Not used in practice
- Many practical methods - often proprietary
  - Step 2 takes many forms
  - Each best for different cases
  - Very great efficiency possible
  - A real art!

Some LP Software

- Several LP methods available for simple problems
  - Solver an Excel tool
  - What's Best http://www.lindo.com (free trials available)
  - GAMS http://www.gams.com/ (big problems)
- They use standard equations and need no special organization
- They provide all kinds of sensitivity information
- The difficult part is setting up the LP equations so that they make sense!

Discussion of Set-up for Solver

- The slide showing a solution has extra information to make slide easy to understand
- The left-hand column (with the equations) and the center columns (with the less or greater than symbols) are not necessary
- See help menu for Solver in Excel

Typical Formulations: "Transportation" Problem

Objective = Minimize cost of moving a single commodity from sources "i" to uses "j"

$= \Sigma_{ij} C_{ij} X_{ij}$

Subject to:

Amount shipped < Amount available $\Sigma_j X_{ij} \leq S_i$

Amount delivered > Uses $\Sigma_i X_{ij} \geq S_j$

Note: Matrix of constraint coefficients are all 0's and 1's ==> Particularly efficient solutions

Typical Formulations: "Blending" or "Diet" Problems

Objective = Minimize cost of materials

$= \Sigma_i C_i X_i$

Subject to:

Limits on availability $X_i \leq$ **Amounts given**

Maxima or minima on impurities, trace elements, nutritional requirements, etc... $\Sigma_i a_{ij} X_i \leq = \geq b_j$

Example:

– **Minimize cost of steel alloy when only so much scrap is available, subject to limitations on carbon content, trace elements, etc.**

Typical Formulations: "Activity" Problems

Objective = Minimize cost of production

= $\Sigma_i C_i X_i$

where X_i represent "activities", that is, specific ways or fixed ratios of using resources

Subject to: Limits on resources $\Sigma_i a_{ij} X_i \leq = \geq b_j$

Example:

– **Minimize cost of delivering cargo where each "activity" represents the use of a different size of ship, each with its own implications for the use of crew, fuel, etc.**

Summary

- **LP can handle very large problems**
- **Basic mathematics simple: User can focus on definition of the problem**
- **Realistic problems require special techniques to deal with non-linearities, integral variables, etc. Can be very sophisticated**
- **This presentation is only an introduction!**