Production Functions (PF)

Outline

1. Motivation
2. Definition
3. Technical Efficiency
4. Mathematical Representation
5. Characteristics

Production Function - Motivation

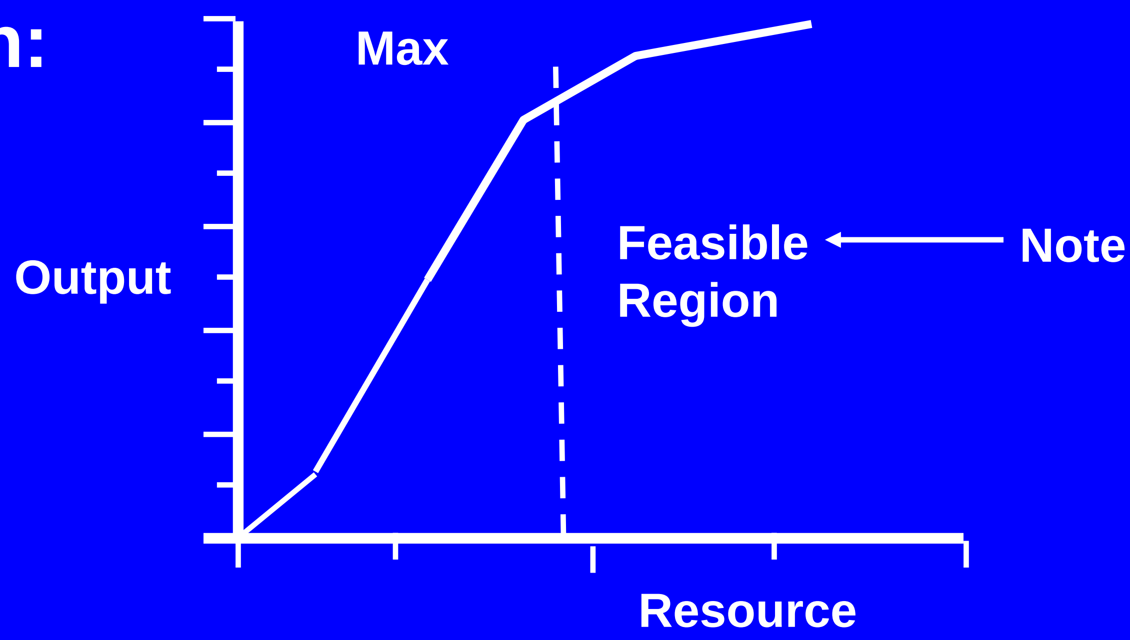
- In order to analyze a system, we need to model it, that is, provide connection between what we do, and what results
- Moreover, we need to focus our attention on the most interesting possibilities…
- This is role of “Production Function”
- Concept derived from Microeconomics
- It is basic Conceptual Structure for Modeling Engineering Systems

Production Function - Definition

- **Definition:**
  - Represents <u>technically efficient</u> transform of physical resources $X = (X_1 \ldots X_n)$ into product or outputs Y (may be good or bad)
- **Example:**
  - Use of aircraft, pilots, fuel (the X factors) to carry cargo, passengers and create pollution (the Y)
- **Typical focus on 1-dimensional output**

Technical Efficiency

- A Process is Technically Efficient if it provides Maximum product from a given set of resources $X = X_1 , \ldots X_n$
- Graph:



Mathematical Representation -- General

- Two Possibilities
- Deductive -- Economic
  - Standard <u>economic</u> analysis
  - Fit data to convenient equation
  - Advantage - ease of use
  - Disadvantage - poor accuracy
- Inductive -- Engineering
  - Standard <u>engineering</u> process
  - Create system model from knowledge of details
  - Advantage - accuracy
  - Disadvantage - careful technical analysis needed

contrast

Mathematical Representation -- Deductive

- Standard Cobb-Douglas Production Function Y $= a_0 \pi X_i^{a_i} = a_0 X_1^{a_1} \ldots X_n^{a_n}$ [π means multiplication]
  - Interpretation: 'a_i' are physically significant
  - Easy estimation by linear least squares
    $\log Y = \log a_0 + \Sigma a_i \log X_i$
- Translog PF -- more recent, less common
  - $\log Y = a_0 + \Sigma a_i \log X_i + \Sigma\Sigma a_{ij} \log X_i \log X_j$
  - Allows for interactive effects
  - More subtle, more realistic
- Economist models (no technical knowledge)

PF Example

- One of the advantage of the “economist” models is that they make calculations easy. This is good for examples, even if not as realistic as Technical Cost Models (next)
- Thus: Output = $2 M^{0.4} N^{0.8}$
- Let’s see what this looks like...

PF Example -- Calculation

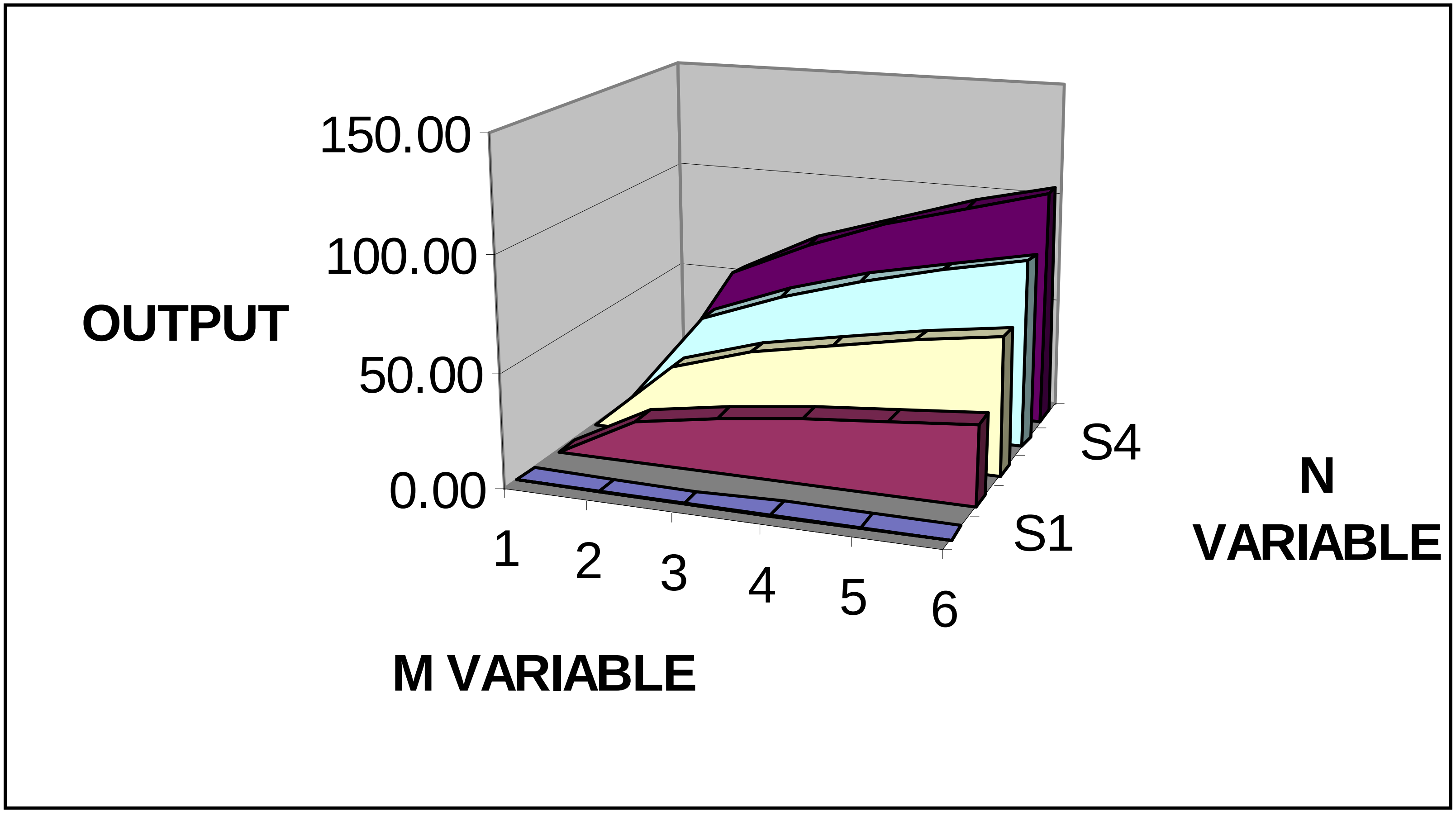
M	N	Output					
"b7"	**"c7"**				**N VARIABLE**		
10	10	31.70	**0**	**5**	**10**	**15**	**20**
		0	0.00	0.00	0.00	0.00	0.00
	M	**10**	0.00	18.21	31.70	43.84	55.19
		20	0.00	24.02	41.83	57.85	72.82
	VARIABLE	**30**	0.00	28.25	49.19	68.04	85.65
		40	0.00	31.70	55.19	76.34	96.09
		50	0.00	34.66	60.34	83.46	105.06

The formula in Excel to calculate the output is: = 2((power(b7,0.4))*(power(c7,0.8))

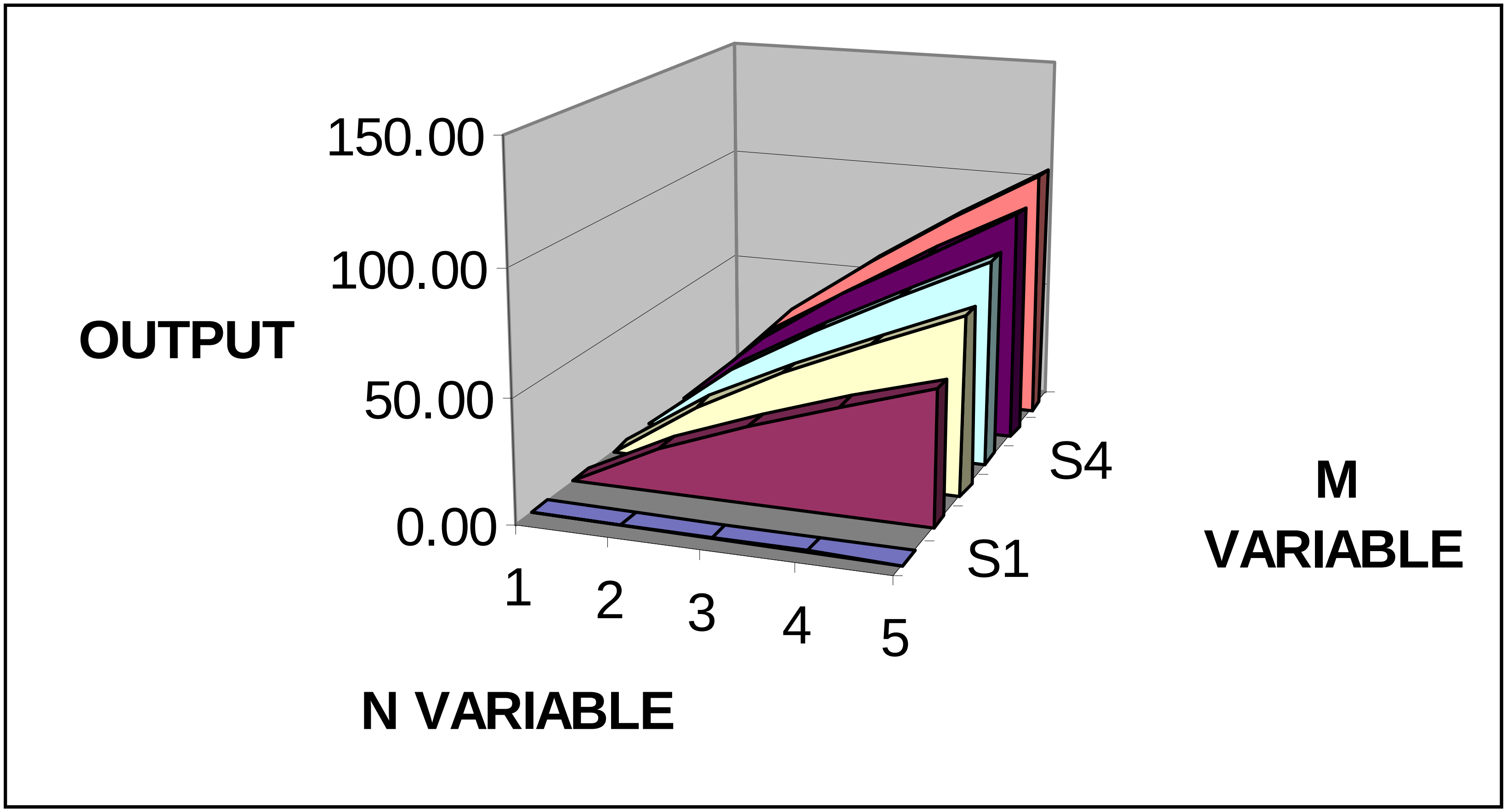
We calculate ouput for many values of the variables using a 2-way Data Table

Recall: Output = 2 $M^{0.4}$ $N^{0.8}$

PF Example -- Graphs



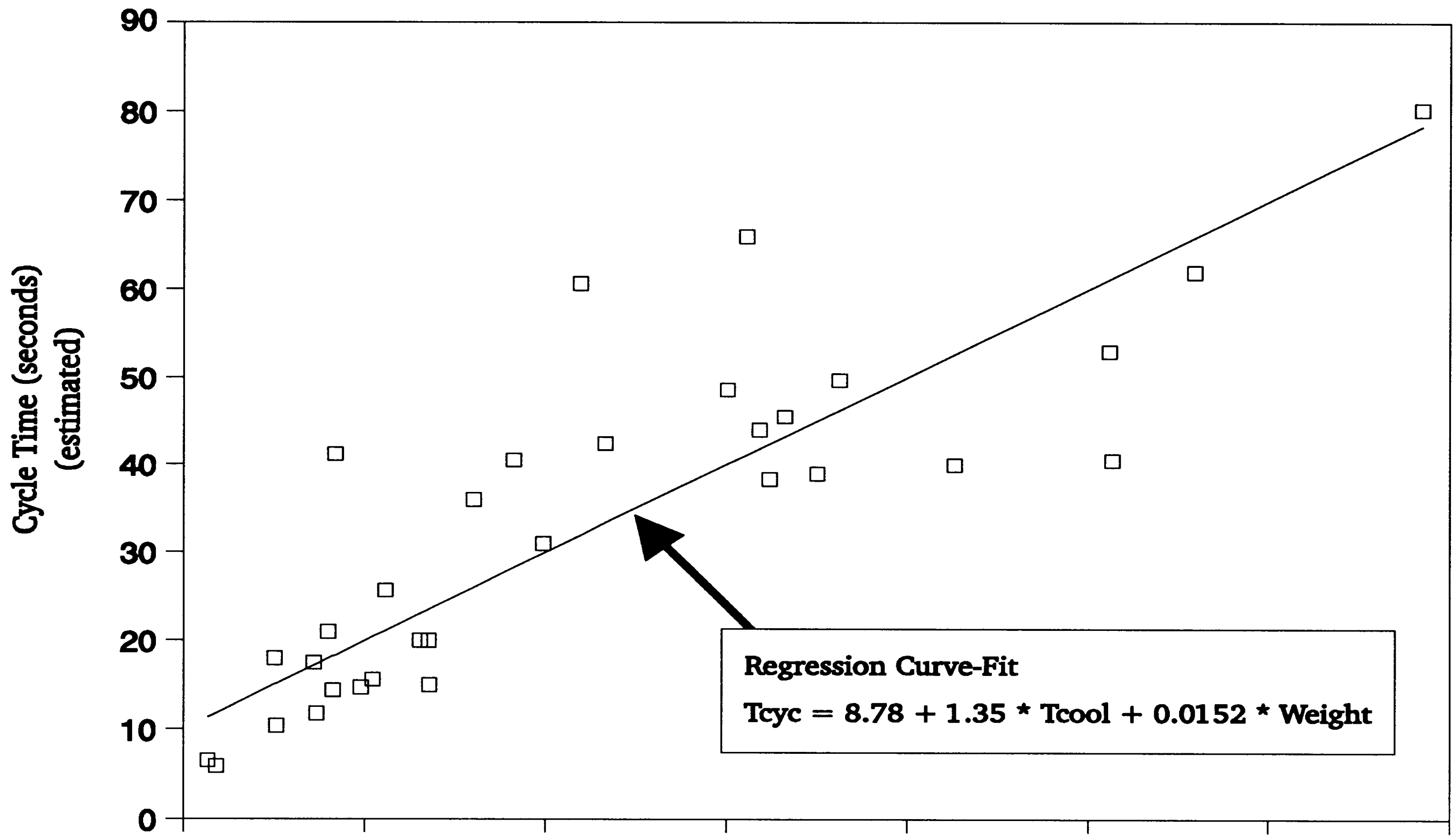
PF Example -- Graphs



Mathematical Representation -- Inductive

- "Engineering models" of PF
- Analytic expressions
  - Rarely applicable: manufacturing is inherently discontinuous
  - Exceptions: process exists in force field, for example transport in fluid, river
- Detailed simulation, Technical Cost Model
  - Generally applicable
  - Requires research, data, effort
  - Wave of future -- not yet standard practice

Cooling Time, Part Weight, Cycle Time Correlation (MIT MSL, Dr.Field)



Common Practice

- We use “models” of our systems all the time – typically computer based
- They are the form of :

  Output = f {design, other parameters}

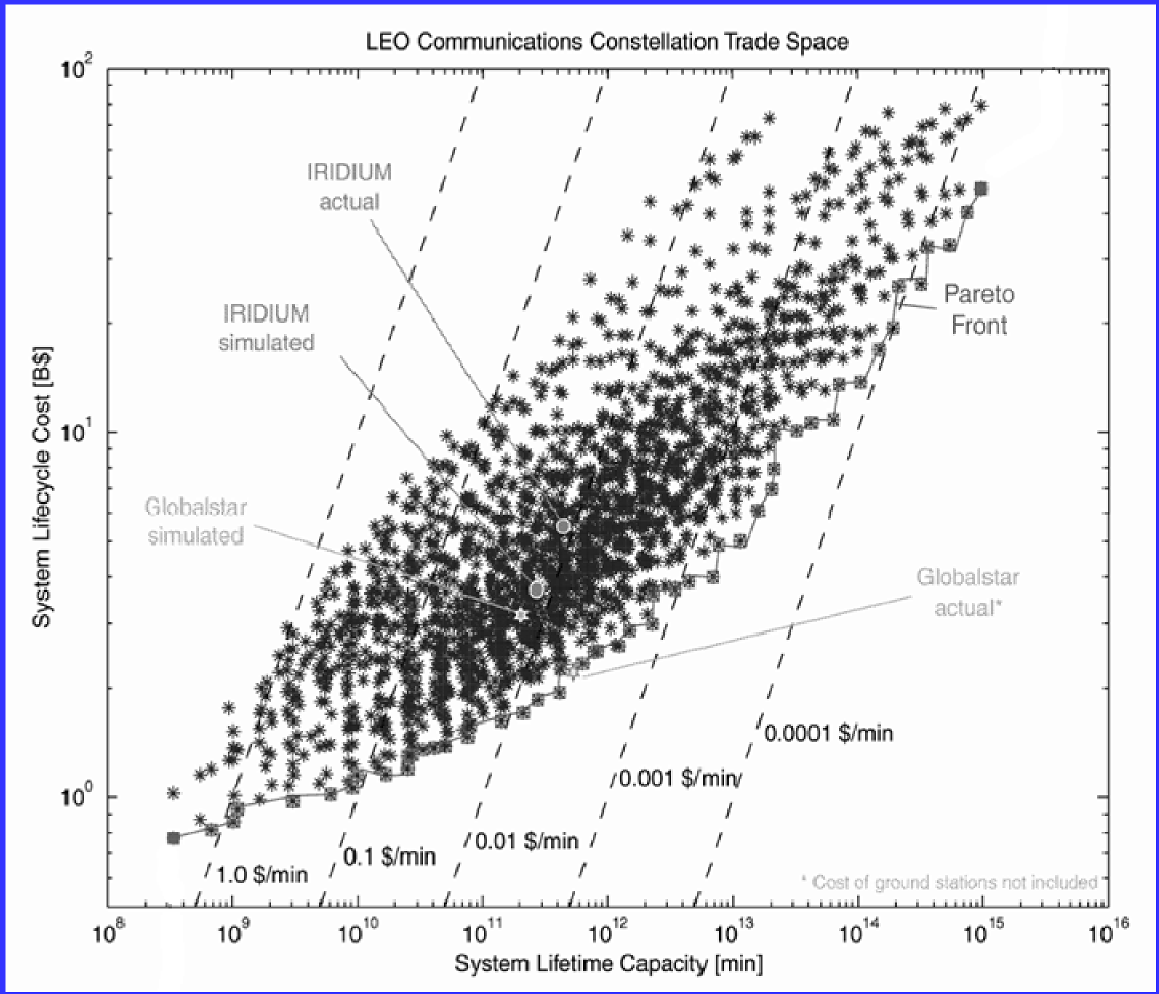
- These generally not Production Functions
- Why?
- Because many designs are technically inefficient, interior to feasible space, PF

Example of Typical Engineering Model

Model Used for Analysis of Telecom Satellites

Lifecycle cost versus Capacity (log scale)

Source: de Weck et al



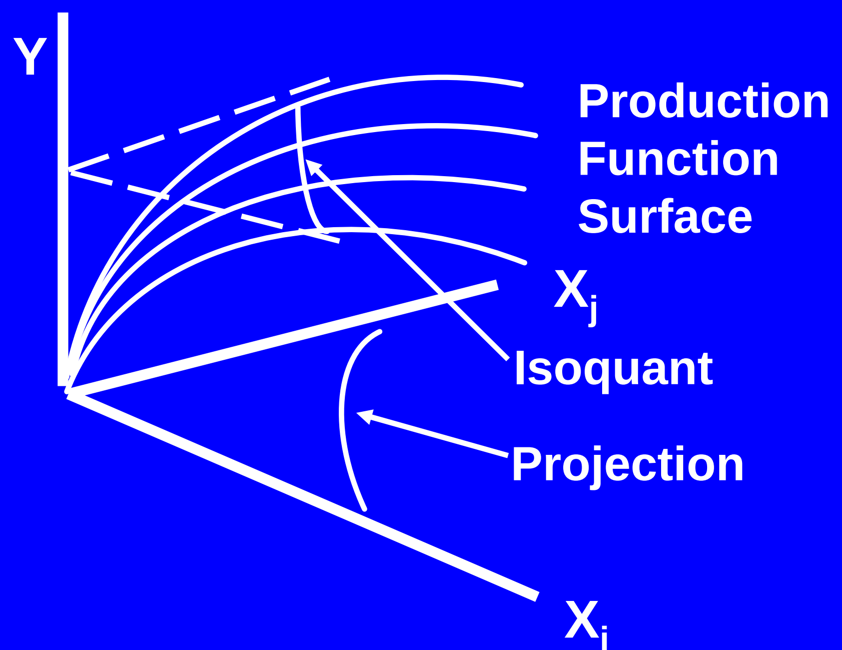
PF: Characteristics

- Isoquants
- Marginal Products
- Marginal Rates of Substitution
- Returns to Scale
- Possible Convexity of Feasible Region

Characteristic: Isoquants

- Isoquant is the Locus (contour) of equal product on production function

- Graph:



Important Implication of Isoquants

- Many designs are technically efficient
  - All points on isoquant are technically efficient
  - no technical basis for choice among them
  - Example:
    - little land, much steel => tall building
    - more land, less steel => low building

- Best System Design depends on Economics
- Values are decisive!

Isoquant Example -- Calculation

For any given output, we can calculate the M value as a function of the N value. Thus: for output = 20, the formula is:

= power(10,2.50)/ (power(c7,2))

= (20/2)exp2.5 / (N exp2)

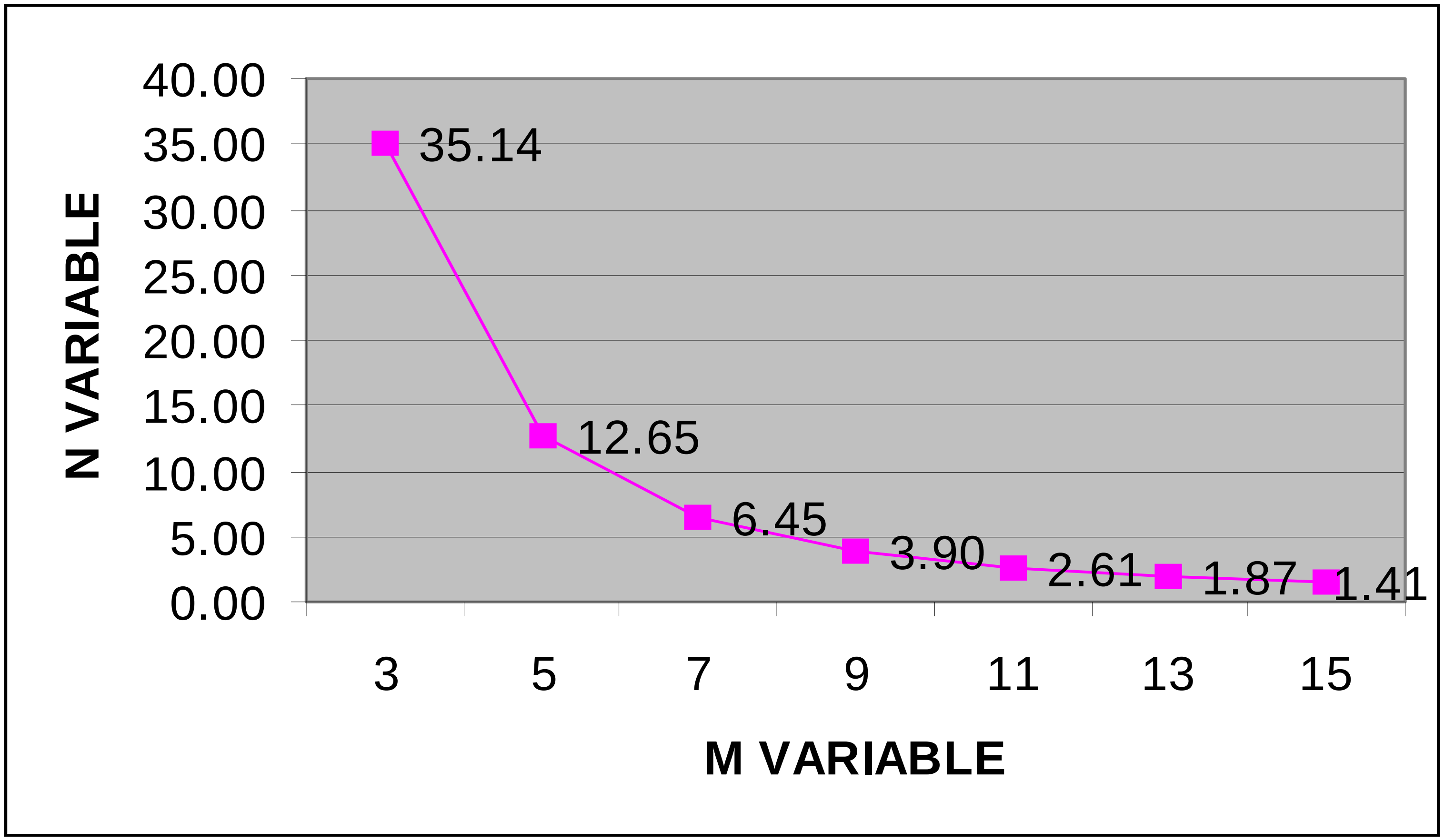
A 1-way data table calculates the (M,N) combinations that constitute the isoquant

M for OUTPUT= 20

N	
c7=10	3.16 ← formula
3	35.14
5	12.65
7	6.45
9	3.90
11	2.61
13	1.87
15	1.41

Recall: Output = $2 M^{0.4} N^{0.8}$

Isoquant Example -- Graph

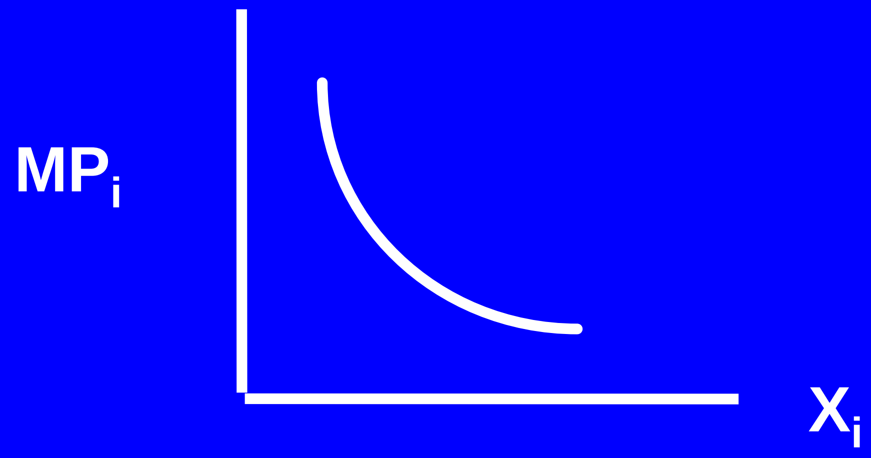


Characteristic: Marginal Products

- Marginal Product is the change in output as only one resource changes

$$MP_i = \partial Y / \partial X_i$$

- Graph:



Diminishing Marginal Products

- **Math:**

$$Y = a_0 X_1^{a_1} \ldots X_i^{a_i} \ldots X_n^{a_n}$$

$$\partial Y / \partial X_i = (a_i / X_i)\, Y \quad = f \; (X_i^{a_i - 1})$$

Diminishing Marginal Product if $a_i < 1.0$

- **"Law" of Diminishing Marginal Products**
  - **Commonly observed -- but not necessary**
  - **"Critical Mass" phenomenon => creates contrary, increasing marginal products**

MP Example -- Calculations

MARGINAL PRODUCT FOR M
(FOR N = 12.65)

C7=10	1.53
3	3.15
5	2.32
7	1.90
9	1.63
11	1.45
13	1.31
15	1.20

The formula for the marginal product is

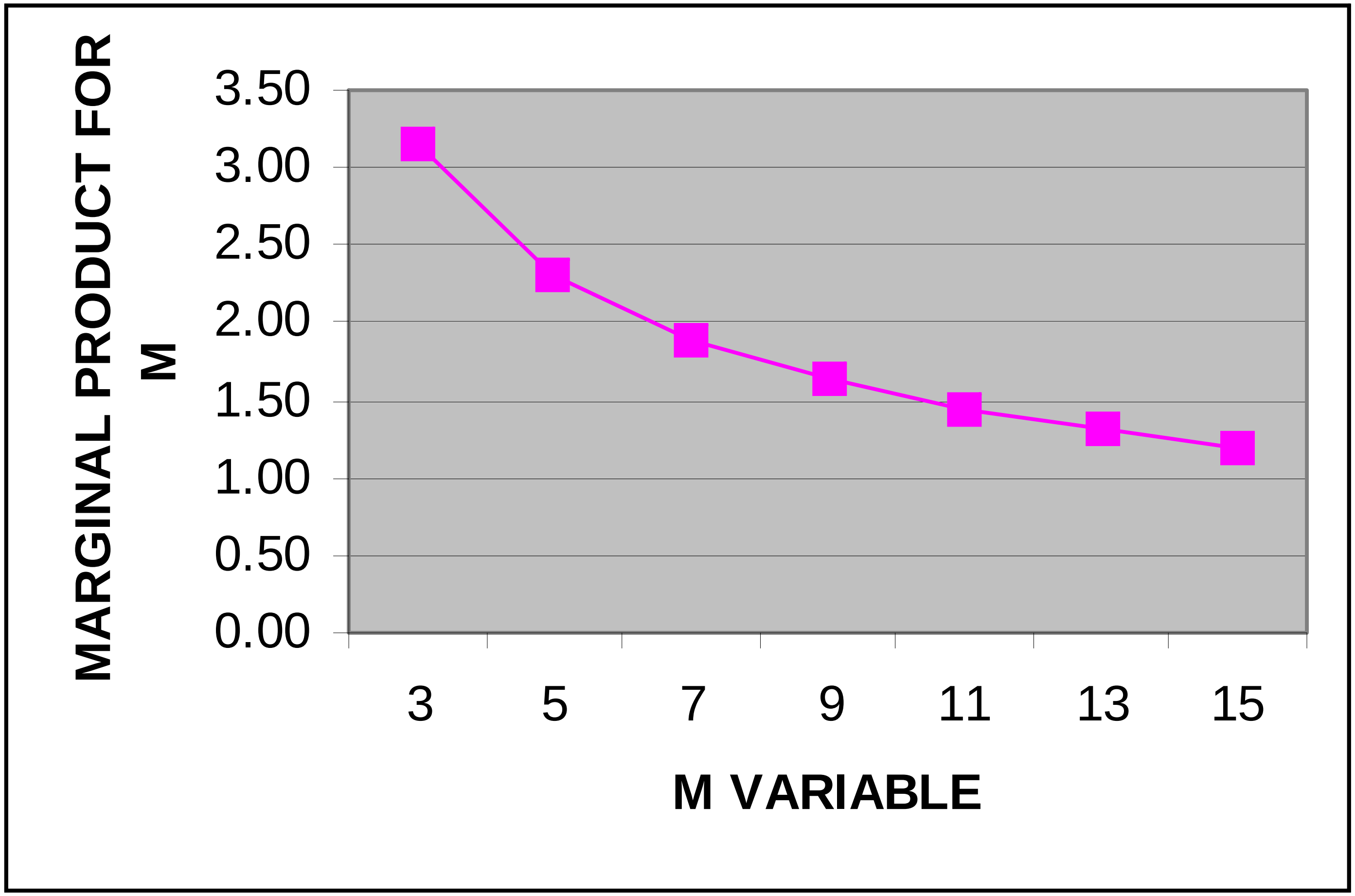
= (0.4/b7)*(2)*(power(b7,0.4))*power(12.65,0,8)

Note that the Marginal Product is conditional on the change in only one variable (in this case M). All other variables are fixed (in this case N=12.65).

Obviously, the Marginal Product depends on the "cut" of the production function you take.

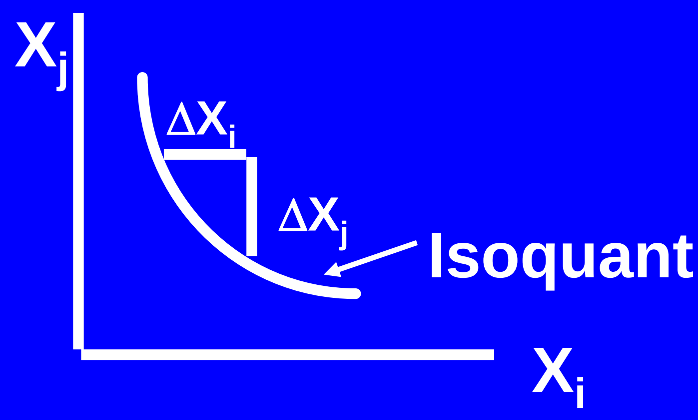
Recall: Output = 2 $M^{0.4} N^{0.8}$

MP Example -- Graph



Characteristic: Marginal Rate of Substitution

- Marginal Rate of Substitution is the Rate at which one resource must substitute for another so that product is constant
- Graph:



Marginal Rate of Substitution (cont'd)

- **Math:**

  since $\Delta X_I MP_I + \Delta X_J MP_J = 0$
  (no change in product)
  then $MRS_{IJ} = \Delta X_J / \Delta X_I$

  $= - MP_I / MP_J = - [(a_I / X_I) Y] / [(a_J / X_J) Y]$

  $= - (a_I / a_J)(X_J / X_I)$

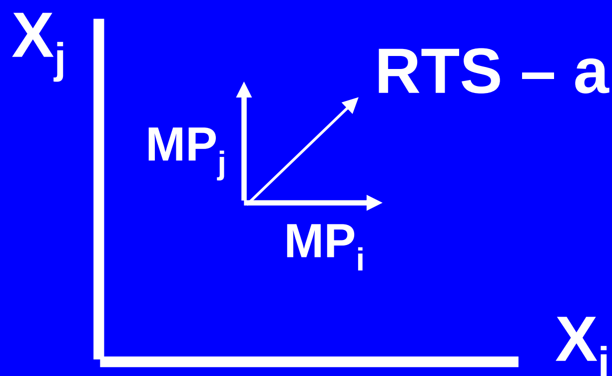
- **MRS is "slope" of isoquant**
  - It is negative
  - Loss in 1 dimension made up by gain in other

MRS Example

- For our example PF: Output = $2 M^{0.4} N^{0.8}$
- $a_M = 0.4$; $a_N = 0.8$
- At a specific point, say M = 5, N = 12.65
- MRS = - (0.4 / 0.8) (12.65 / 5) = - 1.265
- At that point, it takes ~ 5/4 times as much N as M to get the same change in output

Characteristic: Returns to Scale

- Returns to Scale is the Ratio of rate of change in Y to rate of change in ALL X (each X_i changes by same factor)
- Graph:
  - Directions in which the rate of change in output is measured for MP and RTS



RTS – along a ray from origin

Returns to Scale (cont'd)

- **Math:**

$$Y' = a_0 \pi X_i^{a_i}$$

$$Y'' = a_0 \pi (sX_i)^{a_i} = Y'(s)^{\Sigma a}$$ all inputs increase by s

$$RTS = (Y''/Y')/s = s^{(\Sigma a_i - 1)}$$

Y"/Y' = % increase in Y

if Y"/Y' > s => Increasing RTS

Increasing returns to scale (IRTS) if $\Sigma a_i > 1.0$

Increasing RTS Example

- **The PF is: Output = 2 $M^{0.4}$ $N^{0.8}$**
  - **Thus $\Sigma a_i = 0.4 + 0.8 = 1.2 > 1.0$**
  - **So the PF has Increasing Returns to Scale**
  - **Compare outputs for (5,10), (10,20), (20,40)**

				N VARIABLE		
10	31.70	**0**	**5**	**10**	**15**	**20**
	0	0.00	0.00	0.00	0.00	0.00
M	**10**	0.00	**18.21**	31.70	43.84	55.19
	20	0.00	24.02	**41.83**	57.85	72.82
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	40	0.00	31.70	55.19	76.34	**96.09**
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Importance of Increasing RTS

- Increasing RTS means that bigger units are more productive than small ones
- IRTS => concentration of production into larger units
- Examples:
  - Generation of Electric power
  - Chemical, pharmaceutical processes

Practical Occurrence of IRTS

- Frequent!
- Generally where
  - * Product = *f* (volume) and
  - * Resources = *f* (surface)
- Example:
  - * ships, aircraft, rockets
  - * pipelines, cables
  - * chemical plants
  - * etc.

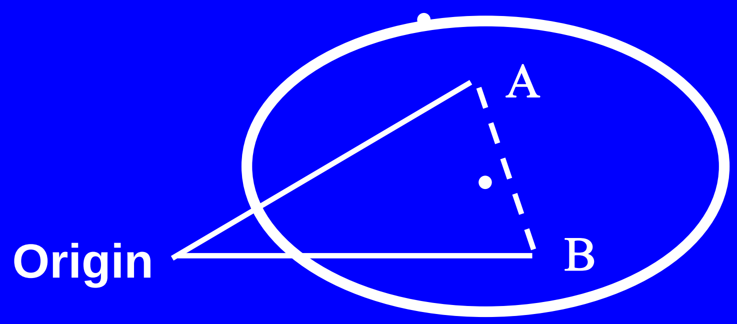
Characteristic: Convexity of Feasible Region

- A region is convex if it has no “reentrant” corners
- Graph:

Informal Test for Convexity of Feasible Region (cont'd)

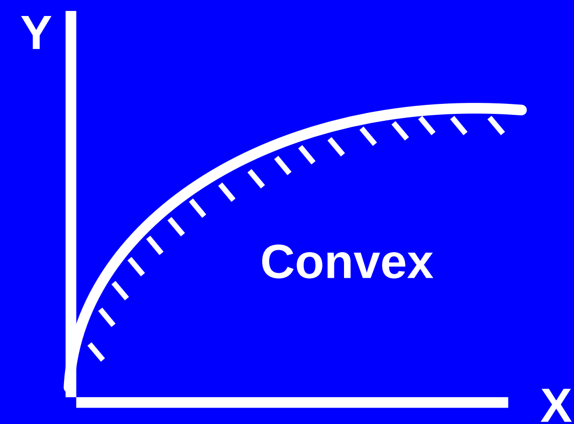
- **Math: If A, B are two vectors to any 2 points in region**

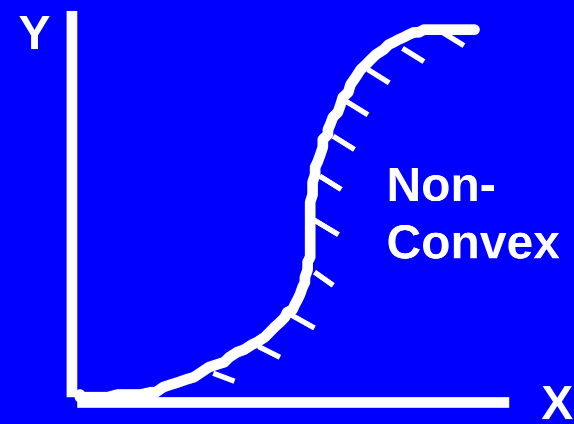
  Convex if all
  $T = KA + (1-K)B \qquad 0 \leq K \leq 1$
  entirely in region



Convexity of Feasible Region for Production Function

- Feasible region of Production function is convex if no reentrant corners

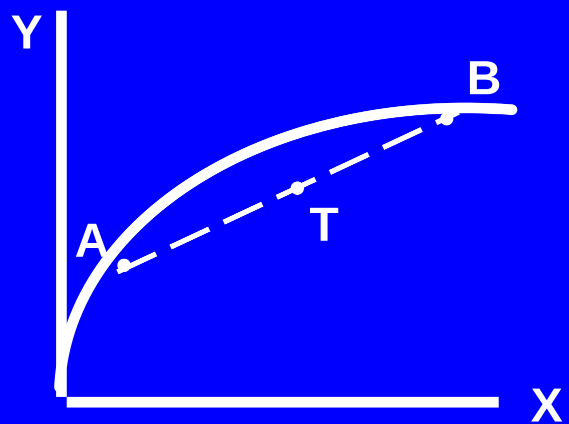


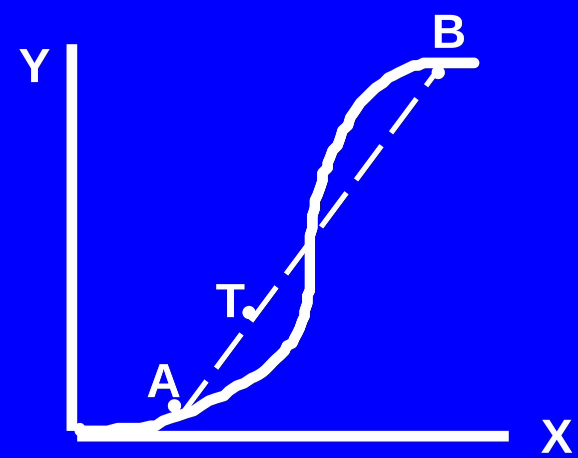


- Convexity => Easier Optimization by linear programming (most common form)
- Non-convex => very difficult optimization

Test for Convexity of Feasible Region of Production Function

- Test for Convexity: Given A,B on PF
  If $T = KA + (1-K)B \quad 0 \leq K \leq 1$
  Convex if all T in region

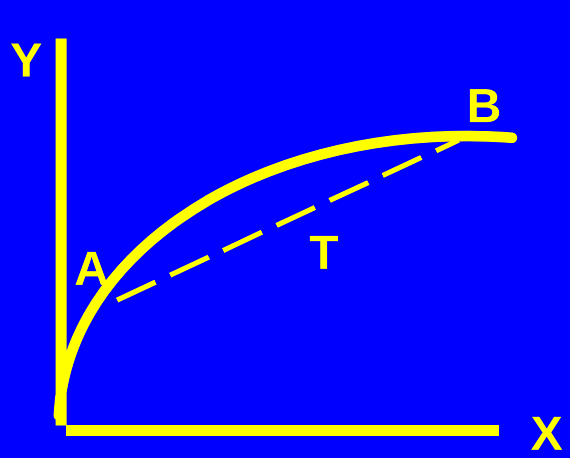




- For Cobb-Douglas, the test is if:
  all $a_i \leq 1.0$ and $\Sigma a_i \leq 1.0$

Convexity Test Example

- Example PF has Diminishing MP, so in the MP direction it looks like left side
- But: it has IRTS, like bottom of right side
- Entire feasible Region is not convex
- However, feasible region for isoquant convex!



Summary

- Production models are the way to describe technically efficient systems

- Important characteristics
  - Isoquants, Marginal products, Marginal rates of Substitution, Returns to scale, possible convexity

- Two ways to represent
  - Economist formulas
  - Technical models (generally more accurate)