Marginal Analysis and Economies of Scale

Purposes:

1. To present a basic application of constrained optimization
2. Apply to Production Function to get criteria and patterns of optimal system design
3. Define Economies of Scale

Marginal Analysis: Outline

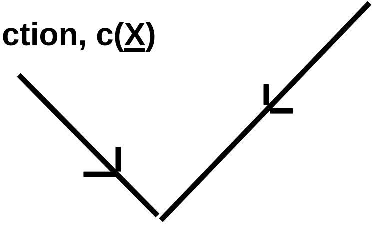
1. Definition and Assumptions
2. Optimality criteria
   - Analysis
   - Interpretation
   - Application
3. Key concepts
   - Expansion path
   - Cost function
   - Economies of scale
4. Summary

Marginal Analysis: Concept

- Basic form of optimization of design
- Combines:

  Production function - Technical efficiency

  Input cost function, c(<u>X</u>)



Economic efficiency

Marginal Analysis: Assumptions

- Feasible region is convex (over relevant portion) This is key. Why?
  - To guarantee no other optimum missed
- No constraints on resources
  - To define a general solution
- Models are “analytic” (continuously differentiable)
  - Finds optimum by looking at margins -- derivatives

Optimality Conditions for Design, by Marginal Analysis

The Problem:

Min $C(Y') = c(\underline{X})$ cost of inputs

for any level of output, Y'

s.t. $g(\underline{X}) = Y'$ production function

vector of resources

Fixed level of output

The Lagrangean:

$$L = c(\underline{X}) - \lambda\,[g(\underline{X}) - Y']$$

Optimality Conditions for Design: Results

- **Key Result:**

$$\partial c(\underline{X}) / \partial X_i = \lambda\, \partial g(\underline{X}) / \partial X_i$$

marginal cost marginal product

- **Optimality Conditions:**

$MP_i / MC_i = MP_j / MC_j = 1 / \lambda$

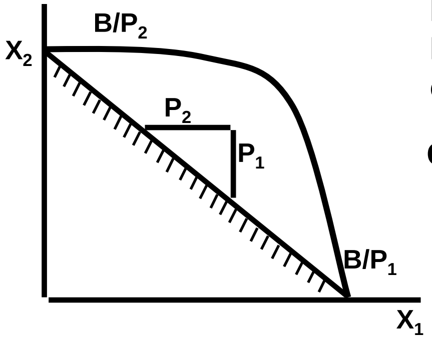
$MC_j / MP_j = MC_i / MP_i = \lambda$ = Shadow Price on Product

- A $\boxed{\text{balanced design}}$

Each X_i contributes "same bang for buck"

Optimality Conditions: Graphical Interpretation of Costs

(A) Input Cost Function



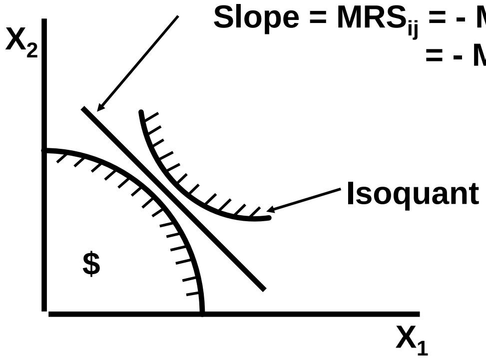
Linear Case:
B = Budget
$c(\underline{X}) = \Sigma p_i X_i \leq B$

General case:
Budget is non-linear
(as in curved line)

Optimality Conditions: Graphical Interpretation of Results

(B) Conditions

Slope = MRS_{ij} = - MP_1 / MP_2
= - MC_1 / MC_2



Application of Optimality Conditions -- Conventional Case

Problem: $Y = a_0 X_1^{a1} X_2^{a2}$

$c(\underline{X}) = \Sigma p_i X_i$

Note: Linearity of Input Cost Function
- typically assumed by economists
- in general, not valid
  - prices rise with demand
  - wholesale, volume discounts

Solution: $[a_1 / X_1^*]\ Y / p_1 = [a_2 / X_2^*]\ Y / p_2$

$==> \quad [a_1 / X_1^*]\ p_1 = [a_2 / X_2^*] / p_2$

(* denotes an optimum value)

Optimality Conditions: Example

Assume $Y = 2X_1^{0.48} X_2^{0.72}$ (increasing RTS)

$c(\underline{X}) = 5X_1 + 3X_2$

Apply Conditions: = MP_i / MC_i

$$[a_1 / X_1^*] / p_1 = [a_2 / X_2^*] / p_2$$

$$[0.48 /5] / X_1^* = [\ 0.72/ 3] / X_2^*$$

$$9.6 / X_1^* = 24 / X_2^*$$

This can be solved for a general relationship between resources => Expansion Path in this case: $X_1^* = 0.4\ X_2^*$

Expansion Path

- Locus of all optimal designs $\underline{X}$*
- Not a property of technical system alone
- Depends on local prices
- Optimal designs do not, in general, maintain constant ratios between optimal X_i*

  Compare: crew of 20,000 ton ship
  crew of 200,000 ton ship

  larger ship does not need 10 times as many sailors (or captains) as smaller ship

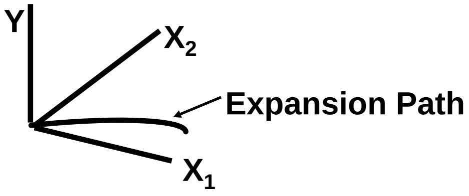
Expansion Path: Non-Linear Prices

- Assume: $Y = 2X_1^{0.48} X_2^{0.72}$ (increasing RTS)
  $c(\underline{X}) = X_1 + X_2^{1.5}$ (increasing costs)
- Optimality Conditions:

$$(0.48 / X_1) Y / 1 = (0.72 / X_2) Y / (1.5X_2^{0.5})$$

$$\Rightarrow X_1^* = (X_2^*)^{1.5}$$

- Graphically:

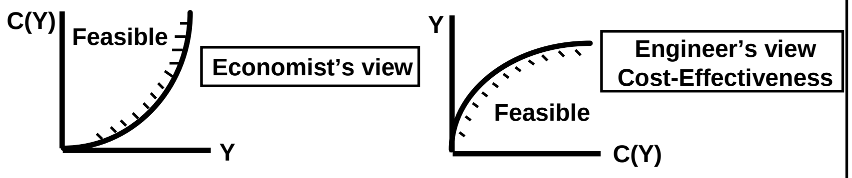


Cost Functions: Concept

- **Not same as input cost function**
  - **It represents the <u>optimal cost of Y</u>**
  - **Not the cost of any set of <u>X</u>**
- **$C(Y) = C(\underline{X}^*) = f\ (Y)$**
- **It is obtained by inserting optimal values of resources (defined by expansion path) into input cost and production functions to give "best cost for any output"**

Cost Functions: Graphical View

- **Graphically:**



- **Great practical use:**
  - **How much Y for budget?**
  - **ΔY for ΔB?**
  - **Cost effectiveness, $\Delta B / \Delta Y$**

Cost Function Calculation: Linear Costs

- Cobb-Douglas Prod. Fcn: $Y = a_0 \pi X_i^{a_i}$
- Linear input cost function: $c(X) = \Sigma p_i X_i$
- Result
  $C(Y) = A(\pi p_i^{a_i/r})Y^{1/r}$ where $r = \Sigma a_i$

- Easy to estimate statistically
  => Solution for ‘a_i’
  => Estimate of prod. fcn. $Y = a_0 \pi X_i^{a_i}$

Cost Function Calculation: Example

- Assume Again: $Y = 2X_1^{0.48} X_2^{0.72}$
  $c(\underline{X}) = X_1 + X_2^{1.5}$
- Expansion Path: $X_1^* = (X_2^*)^{1.5}$

- Thus: $Y = 2(X_2^*)^{1.44}$
  $c(\underline{X}^*) = 2(X_2^*)^{1.5}$
  => $X_2^* = (Y/2)^{0.7}$
  $c(Y) = c(\underline{X}^*) = (2^{-0.05})Y^{1.05}$

Real Example: Communications Satellites

- System Design for Satellites at various altitudes and configurations
- Source: O. de Weck and MIT co-workers
- Data generated by a technical model that costs out wide variety of possible designs
- Establishes a Cost Function
- Note that we cannot in general expand along cost frontier. Technology limits what we can do: Only certain pathways are available

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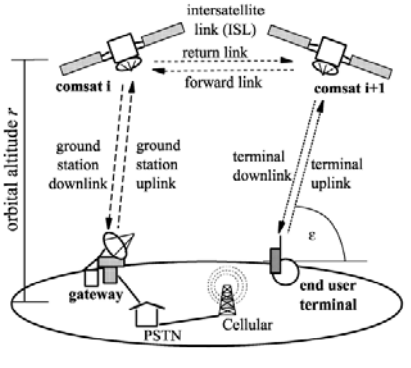
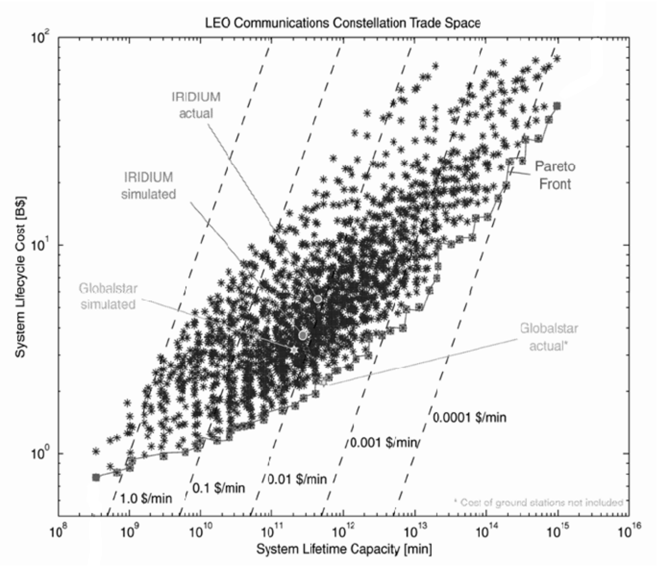
Key Parameters

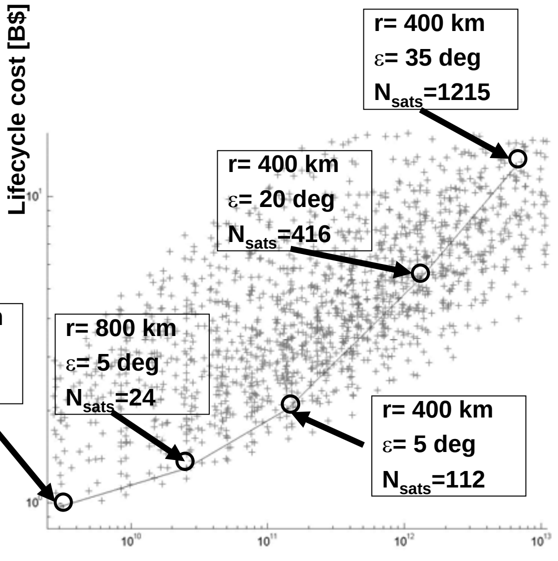
- Each star in the Trade Space corresponds to a vector:
  - ⁎ r: altitude of the satellites
  - – ε: minimum acceptable elevation angle
  - ⁎ C: constellation type
  - ⁎ Pt: maximal transmitter power
  - ⁎ DA: Antenna diameter
  - – Δf_c: bandwidth
  - ⁎ ISL: intersatellite links
  - ⁎ T_{sat}: Satellites lifetime
- Some are fixed:
  - ⁎ C: polar
  - – Δf_c: 40 kHz



Trade Space



Path example



r= 400 km
ε= 35 deg
N_{sats}=1215

r= 400 km
ε= 20 deg
N_{sats}=416

r= 2000 km
ε= 5 deg
N_{sats}=24

r= 800 km
ε= 5 deg
N_{sats}=24

r= 400 km
ε= 5 deg
N_{sats}=112

Constants:

Pt=200 W

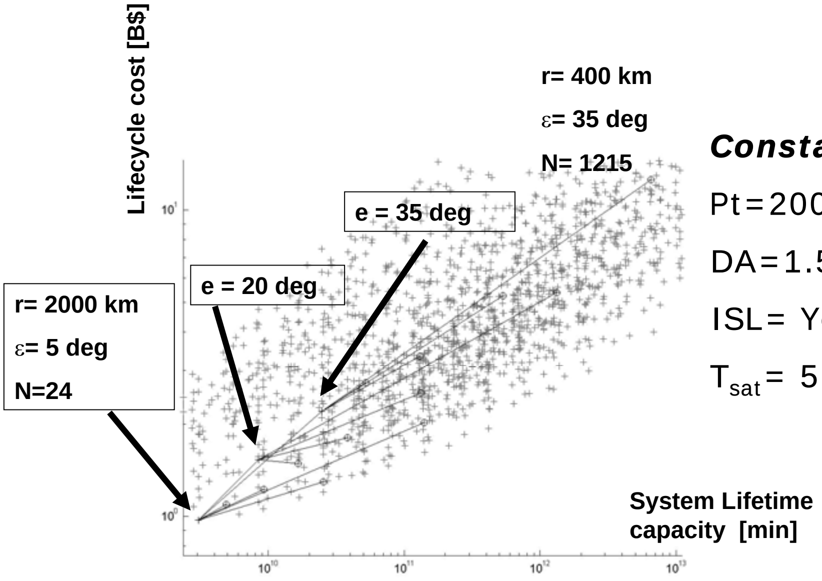
DA=1.5 m

ISL= Yes

T_{sat}= 5

System Lifetime capacity [min]

Tree example



Constants:

Pt=200 W

DA=1.5 m

ISL= Yes

T_{sat}= 5

Economies of Scale: Concept

- **A possible characteristic of cost function**
- **Concept similar to returns to scale, <u>except</u>**
  - **ratio of ‘X_i’ not constant**
  - **refers to costs (economies) not “returns”**
  - **Not universal (as RTS) but depends on local costs**
- **Economies of scale exist if costs increase slower than product**

  Total cost = C(Y) = $Y^{\propto}$ $\propto < 1.0$

EoS: Specific Cases

- If Cobb-Douglas, linear input costs, Increasing RTS ($r = \Sigma a_i > 1.0$)

  => Economies of scale

Optimality Conditions: $[a_1/X_1^*] / p_1 = [a_2/X_2^*] / p_2$

Thus: Inputs Cost is a function of X_1^*

Also: Output is a function of $[X_1^*]^r$

So: X_1^* is a function of $Y^{1/r}$

Finally: $C(Y)$ = function of $[Y^{1/r}]$

EoS: General Case

Not necessarily true in general that Returns to scale => Economies of Scale

Increasing costs may counteract advantages of returns to scale

See example!!

$c(Y) = c(\underline{X}^*) = (2^{-0.05})Y^{1.05}$

Contrarily, if unit prices decrease with volume (quite common) we can have Economies of Scale, without Returns to Scale

EoS: Why important?

Because…Pressure to build large

Consider: $C = aY^{0.7}$ (EoS ?)

Cost per Unit? $= C/Y = a / Y^{0.3}$

If C/Y decreases as scale increases, better to build large…

This is common understanding

Is this right?

EoS: Caution about Conclusion

General cost of Production for a specific plant

Total Cost = Capital + Variable Costs = K + VY

Cost per Unit? = C/Y = K/Y + V

What happens if plant is not operating at full capacity?

Unit costs can be very high for larger plant , higher than for a smaller plant

EoS: Example of previous

Consider two plants, one small one large

		Unit Cost		
Capacity	Total Cost	at Capacity	V = 100	V= 50
100	1000+5V	15	15	25
200	1800+4V	13	22	40

Large plant has EoS…. (Why?)

But for small volumes of production, smaller plants lead to lower unit costs…

Summary:
Marginal Analysis and EoS:

- **Assumptions --**
  - **-- Convex Feasible Region**
  - **-- Unconstrained**
- **Optimality Criteria**
  - **⁎ MC/MP same for all inputs**
- **Expansion path -- Locus of Optimal Design**
- **Cost function -- cost along Expansion Path**
- **Economies of scale (vs Returns to Scale)**
  - **-- Exist if Cost/Unit decreases with volume**
  - **-- Pressure to build large … BUT**