

Constrained Optimization

- **Objective of Presentation:** To introduce Lagrangean as a basic conceptual method used to optimize design in real situations
- **Essential Reality:** In practical situations, the designers are constrained or limited by
 - physical realities
 - design standards
 - laws and regulations, etc.

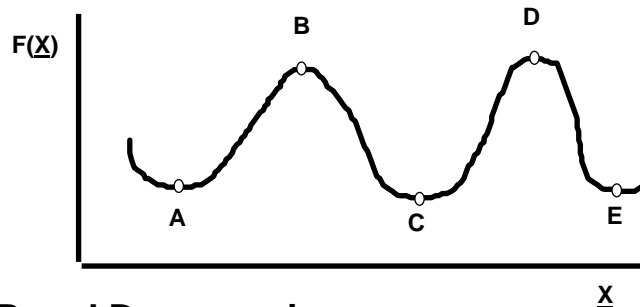
Outline

- **Unconstrained Optimization (Review)**
- **Constrained Optimization – Lagrangeans**
 - Approach
 - Lagrangeans as Equality constraints
- **Interpretation of Lagrangeans as “Shadow prices”**

Unconstrained Optimization: Definitions

- Optimization => Maximum of desired quantity,
or => Minimum of undesired quantity
- Objective Function = Formula to be optimized
= $Z(\underline{X})$
- Decision Variables = Variables about which
we can make decisions
= $\underline{X} = (X_1, \dots, X_n)$

Unconstrained Optimization: Graph



- B and D are maxima
- A, C and E are minima

Unconstrained Optimization: Conditions

- By calculus: if $F(X)$ continuous, analytic
- Primary conditions for maxima and minima:
 $\partial F(X) / \partial X_i = 0 \quad \forall_i$
(symbol means: "for all i")
- Secondary conditions:
 $\partial^2 F(X) / \partial X_i^2 < 0 \quad \Rightarrow \text{Max} \quad (\text{B,D})$
 $\partial^2 F(X) / \partial X_i^2 > 0 \quad \Rightarrow \text{Min} \quad (\text{A,C,E})$
These define whether point of no change in Z is a maximum or a minimum

Unconstrained Optimization: Example

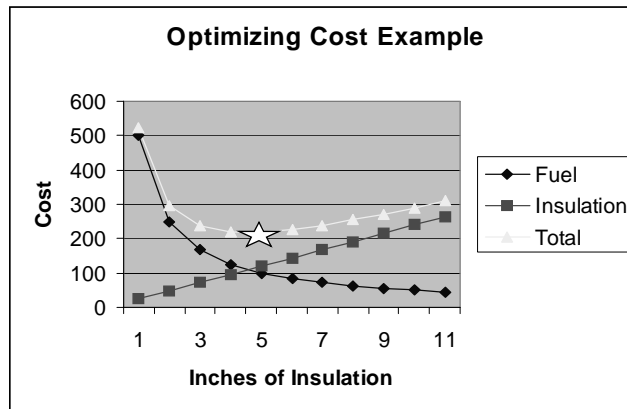
- Example: Housing insulation
Total Cost = Fuel cost + Insulation cost
 x = Thickness of insulation
 $F(x) = K_1 / x + K_2 x$

Primary condition: $\partial F(x) / \partial x = 0 = -K_1 / x^2 + K_2$

 $\Rightarrow x^* = \{K_1 / K_2\}^{1/2}$
(starred quantities are optimal)

Unconstrained Optimization: Graph of Solution to Example

- If: $K_1 = 500$; $K_2 = 24$ Then: $X^* = 4.56$



Engineering Systems Analysis for Design
Massachusetts Institute of Technology

Richard de Neufville ©
Constrained Optimization

Slide 7 of 22

Constrained Optimization: General

- “Constrained Optimization” involves the optimization of a process subject to constraints
- Constraints have two basic types
 - Equality Constraints -- some factors have to equal constraints
 - Inequality Constraints -- some factors have to be less less or greater than the constraints (these are “upper” and “lower” bounds)

Engineering Systems Analysis for Design
Massachusetts Institute of Technology

Richard de Neufville ©
Constrained Optimization

Slide 8 of 22

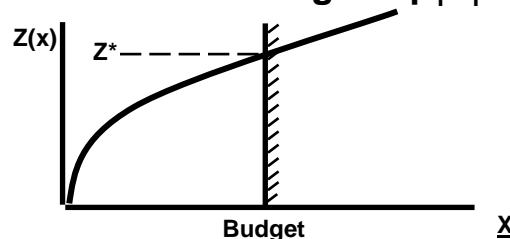
Constrained Optimization: General Approach

- To solve situations of increasing complexity, (for example, those with equality, inequality constraints) ...
- Transform more difficult situation into one we know how to deal with
- Note: this process introduces new variables!

- Thus, transform
 - “constrained” optimization to “unconstrained” optimization

Equality Constraints: Example

- Example: Best use of budget
- Maximize: Output = $Z(\underline{X}) = a_0 x_1^{a_1} x_2^{a_2}$
- Subject to (s.t.):
Total costs = Budget = $p_1 x_1 + p_2 x_2$



Note: $\partial Z(\underline{X}) / \partial \underline{X} \neq 0$ at optimum

Lagrangian Method: Approach

- Transforms equality constraints into unconstrained problem
- Start with:
Opt: $F(\underline{x})$
s.t.: $g_j(\underline{x}) = b_j \Rightarrow g_j(\underline{x}) - b_j = 0$
- Get to: $L = F(\underline{x}) - \sum_j \lambda_j [g_j(\underline{x}) - b_j]$
 $\lambda_j =$ Lagrangean multipliers (lambdas sub j) --
are unknown quantities for which we must solve

Note: $[g_j(\underline{x}) - b_j] = 0$ by definition, thus
optimum for $F(\underline{x}) =$ optimum for L

Lagrangian: Optimality Conditions

- Since the new formulation is a single equation, we can use formulas for unconstrained optimization.
- We set partial derivatives equal to zero for all unknowns, the X and the λ

- Thus, to optimize L :
$$\frac{\partial L}{\partial x_i} = 0 \quad \forall_i$$
$$\frac{\partial L}{\partial \lambda_j} = 0 \quad \forall_j$$

Lagrangean: Example Formulation

- Problem:

$$\text{Opt: } F(\underline{x}) = 6x_1x_2$$

$$\text{s.t.: } g(\underline{x}) = 3x_1 + 4x_2 = 18$$

- Lagrangean:

$$L = 6x_1x_2 - \lambda(3x_1 + 4x_2 - 18)$$

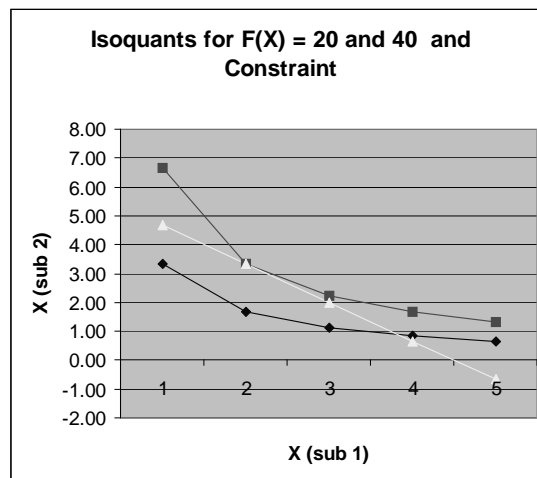
Optimality Conditions:

$$\partial L / \partial x_1 = 6x_2 - 3\lambda = 0$$

$$\partial L / \partial x_2 = 6x_1 - 4\lambda = 0$$

$$\partial L / \partial \lambda_j = 3x_1 + 4x_2 - 18 = 0$$

Lagrangean: Graph for Example



Lagrangean: Example Solution

- Solving as unconstrained problem:

$$\partial L / \partial x_1 = 6x_2 - 3\lambda = 0$$

$$\partial L / \partial x_2 = 6x_1 - 4\lambda = 0$$

$$\partial L / \partial \lambda_i = 3x_1 + 4x_2 - 18 = 0$$

- so that: $\lambda = 2x_2 = 1.5x_1$ (first 2 equations)

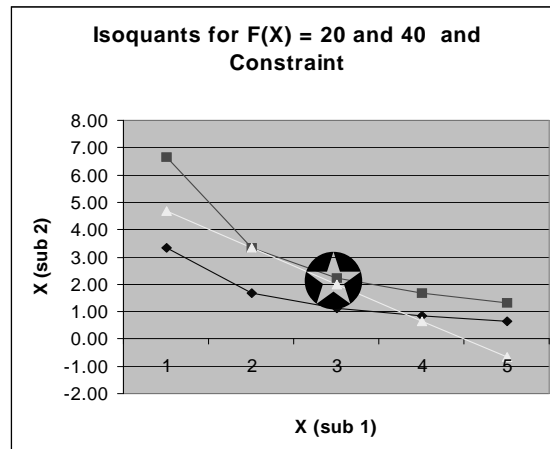
$$\Rightarrow x_2 = 0.75x_1$$

$$\Rightarrow 3x_1 + 3x_1 - 18 = 0 \quad (3\text{rd equation})$$

- $x_1^* = 18/6 = 3 \quad x_2^* = 18/8 = 2.25 \quad \lambda^* = 4.5$

- $F(x)^* = 40.5$

Lagrangean: Graph for Solution



Shadow Price

- **Shadow Price = Rate of change of objective function per unit change of constraint**
$$= \partial F(\underline{x}) / \partial b_j = \lambda_j$$
- **It is extremely important for system design**
- **It defines value of changing constraints, and indicates if worthwhile to change them**
 - **Should we buy more resources?**
 - **Should we change environmental constraints?**

Lagrangean Multiplier is a Shadow Price

- **The Lagrangean multiplier is interpreted as the shadow price on constraint**
- $$\begin{aligned} SP_j &= \partial F(\underline{x})^* / \partial b_j = \partial L^* / \partial b_j \\ &= \partial \{ F(\underline{x}) - \sum_j \lambda_j [g_j(\underline{x}) - b_j] \} / \partial b_j \\ &= \lambda_j \end{aligned}$$
 - **Naturally, this is an instantaneous rate**

Lagrangean = Shadow Prices Example

- Let's see how this works in example, by changing constraint by 0.1 units:

$$\text{Opt: } F(\underline{x}) = 6x_1x_2$$

$$\text{s.t.: } g(\underline{x}) = 3x_1 + 4x_2 = 18.1$$

- The optimum values of the variables are
 $x_1^* = (18.1)/6$ $x_2^* = (18.1)/8$ $\lambda^* = 4.5$
- Thus $F(x)^* = 6(18.1/6)(18.1/8) = 40.95$

$$\Delta F(x) = 40.95 - 40.5 = 0.45 = \lambda^* (0.1)$$

Generalization

- In general constraints are “inequalities”:
 - Upper bounds: $g_j(\underline{x}) \leq b_j$
 - Lower bounds: $g_j(\underline{x}) \geq b_j$
- At optimum, some constraints will limit solution (they are “binding”) others not
 - Example: airline bags: weight $\leq 40\text{kg}$; sum of dimensions $\leq 2.5\text{m}$. Your bag might be limited by weight, not by size.
- Shadow prices
 - ≥ 0 for all “binding” constraints
 - $= 0$ for all others

Design Implications

- Expanding range of design variables (x), increases freedom to improve design, thus adds value
- This is called “relaxing” the constraints
 - Increasing upper bounds
 - Decreasing lower bounds
- As any constraint relaxed, it may no longer be “binding” , and others can become so
- **SHADOW PRICES DEPEND ON OTHER CONSTRAINTS, “PROBLEM DEPENDENT”**

Take-aways

- Relaxing design constraints adds value (in terms of better performance, $F(x)$)
- This value is the “shadow price” of that constraint
- Knowing this can be very important for designers, shows way to improve quickly
- **NOTE: Value to design has no direct connection to cost of constraint, not a “price in ordinary terms**