

Marginal Analysis

Purposes:

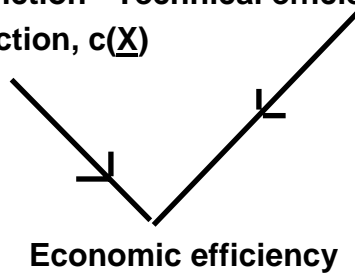
- 1. To present a basic application of constrained optimization**
- 2. Apply to Production Function to get criteria and patterns of optimal system design**

Marginal Analysis: Outline

- 1. Definition and Assumptions**
- 2. Optimality criteria**
 - * Analysis
 - * Interpretation
 - * Application
- 3. Key concepts**
 - * Expansion path
 - * Cost function
 - * Economies of scale
- 4. Summary**

Marginal Analysis: Concept

- Basic form of optimization of design
- Combines:
 - Production function - Technical efficiency
 - Input cost function, $c(\underline{X})$



Marginal Analysis: Assumptions

- Feasible region is convex (over relevant portion) This is key. Why?
 - * To guarantee no other optimum missed
- No constraints on resources
 - * To define a general solution
- Models are “analytic” (continuously differentiable)
 - * Finds optimum by looking at margins -- derivatives

Optimality Conditions for Design, by Marginal Analysis

The Problem:

Min $C(\underline{X}) = c(\underline{X})$ cost of inputs
for any level of output, Y'

s.t. $g(\underline{X}) = Y'$ production function

\swarrow \swarrow
 vector Fixed level of output
 of resources

The Lagrangean:

$$L = c(\underline{X}) - \lambda [g(\underline{X}) - Y']$$

Optimality Conditions for Design: Results

• Key Result:

$$\frac{\partial c(\underline{X})}{\partial X_i} = \lambda \frac{\partial g(\underline{X})}{\partial X_i}$$

\uparrow \uparrow
 marginal cost marginal product

• Optimality Conditions:

$$MP_i / MC_i = MP_j / MC_j = 1 / \lambda$$

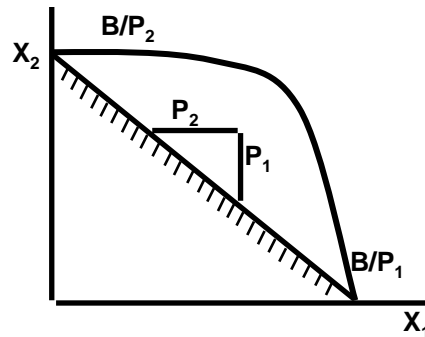
$$MC_j / MP_j = MC_i / MP_i = \lambda = \text{Shadow Price on Product}$$

• A balanced design

Each X_i contributes “same bang for buck”

Optimality Conditions: Graphical Interpretation of Costs

(A) Input Cost Function

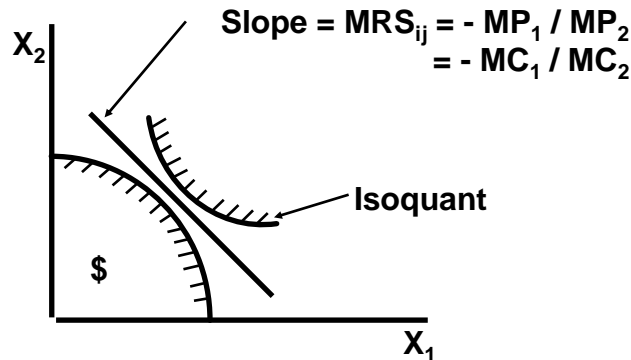


Linear Case:
 $B = \text{Budget}$
 $c(\underline{X}) = \sum p_i X_i \leq B$

General case:
 Budget is non-linear
 (as in curved line)

Optimality Conditions: Graphical Interpretation of Results

(B) Conditions



Application of Optimality Conditions -- Conventional Case

Problem: $Y = a_0 X_1^{a_1} X_2^{a_2}$
 $c(\underline{X}) = \sum p_i X_i$

Note: Linearity of Input Cost Function
 - typically assumed by economists
 - in general, not valid

- prices rise with demand
- wholesale, volume discounts

Solution: $[a_1 / X_1^*] Y / p_1 = [a_2 / X_2^*] Y / p_2$
 $\implies [a_1 / X_1^*] p_1 = [a_2 / X_2^*] / p_2$

(* denotes an optimum value)

Optimality Conditions: Example

Assume $Y = 2X_1^{0.48} X_2^{0.72}$ (increasing RTS)
 $c(\underline{X}) = 5X_1 + 3X_2$

Apply Conditions: = MP_i / MC_i

$$[a_1 / X_1^*] / p_1 = [a_2 / X_2^*] / p_2$$

$$[0.48 / 5] / X_1^* = [0.72 / 3] / X_2^*$$

$$9.6 / X_1^* = 24 / X_2^*$$

This can be solved for a general relationship between resources => Expansion Path in this case: $X_1^* = 0.4 X_2^*$

Expansion Path

- Locus of all optimal designs \underline{X}^*
- Not a property of technical system alone
- Depends on local prices
- Optimal designs do not, in general, maintain constant ratios between optimal X_i^*

Compare: crew of 20,000 ton ship
 crew of 200,000 ton ship

larger ship does not need 10 times as many
 sailors (or captains) as smaller ship

Expansion Path: Non-Linear Prices

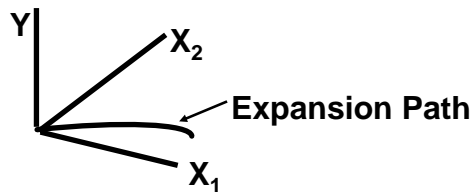
- Assume: $Y = 2X_1^{0.48} X_2^{0.72}$ (increasing RTS)
 $c(\underline{X}) = X_1 + X_2^{1.5}$ (increasing costs)

- Optimality Conditions:

$$(0.48 / X_1) Y / 1 = (0.72 / X_2) Y / (1.5X_2^{0.5})$$

$$\Rightarrow X_1^* = (X_2^*)^{1.5}$$

- Graphically:

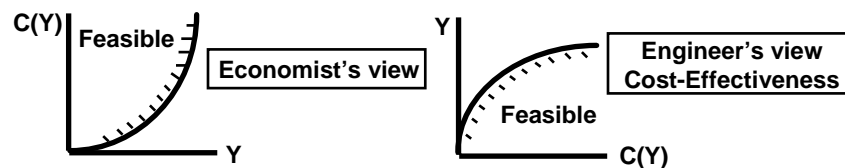


Cost Functions: Concept

- **Not same as input cost function**
It represents the optimal cost of Y
Not the cost of any set of X
- $C(Y) = C(\underline{X}^*) = f(Y)$
- It is obtained by inserting optimal values of resources (defined by expansion path) into input cost and production functions to give “best cost for any output”

Cost Functions: Graphical View

- **Graphically:**



- **Great practical use:**
How much Y for budget?
 ΔY for ΔB ?
Cost effectiveness, $\Delta B / \Delta Y$

Cost Function Calculation: Linear Costs

- Cobb-Douglas Prod. Fcn: $Y = a_0 \pi X_i^{a_i}$
- Linear input cost function: $c(X) = \sum p_i X_i$
- Result
$$C(Y) = A(\pi p_i^{a_i/r}) Y^{1/r} \quad \text{where } r = \sum a_i$$
- Easy to estimate statistically
=> Solution for 'a_i'
=> Estimate of prod. fcn. $Y = a_0 \pi X_i^{a_i}$

Cost Function Calculation: Example

- Assume Again: $Y = 2X_1^{0.48} X_2^{0.72}$
$$c(\underline{X}) = X_1 + X_2^{1.5}$$
- Expansion Path: $X_1^* = (X_2^*)^{1.5}$
- Thus: $Y = 2(X_2^*)^{1.44}$
$$c(\underline{X}^*) = 2(X_2^*)^{1.5}$$

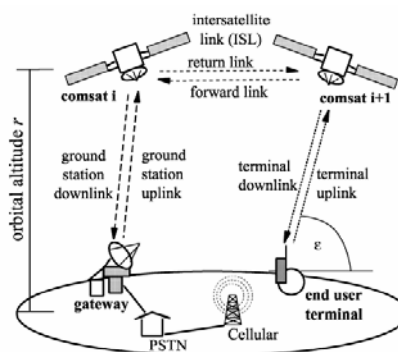
=> $X_2^* = (Y/2)^{0.7}$
$$c(Y) = c(\underline{X}^*) = (2^{-0.05}) Y^{1.05}$$

Real Example: Communications Satellites

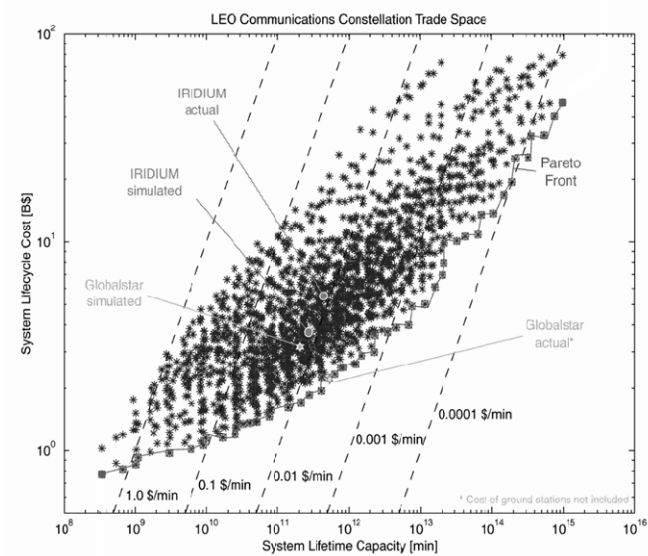
- System Design for Satellites at various altitudes and configurations
- Source: O. de Weck and MIT co-workers
- Data generated by a technical model that costs out wide variety of possible designs
- Establishes a Cost Function
- Note that we cannot in general expand along cost frontier. Technology limits what we can do: Only certain pathways are available

Key Parameters

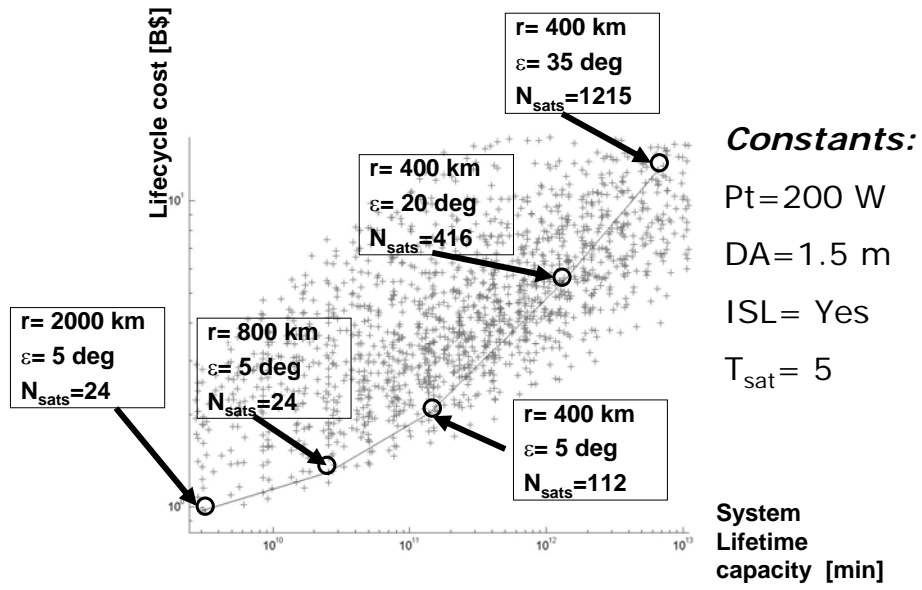
- Each star in the Trade Space corresponds to a vector:
 - * r : altitude of the satellites
 - ϵ : minimum acceptable elevation angle
 - * C : constellation type
 - * P_t : maximal transmitter power
 - * DA : Antenna diameter
 - Δf_c : bandwidth
 - * ISL: intersatellite links
 - * T_{sat} : Satellites lifetime
- Some are fixed:
 - * C : polar
 - Δf_c : 40 kHz



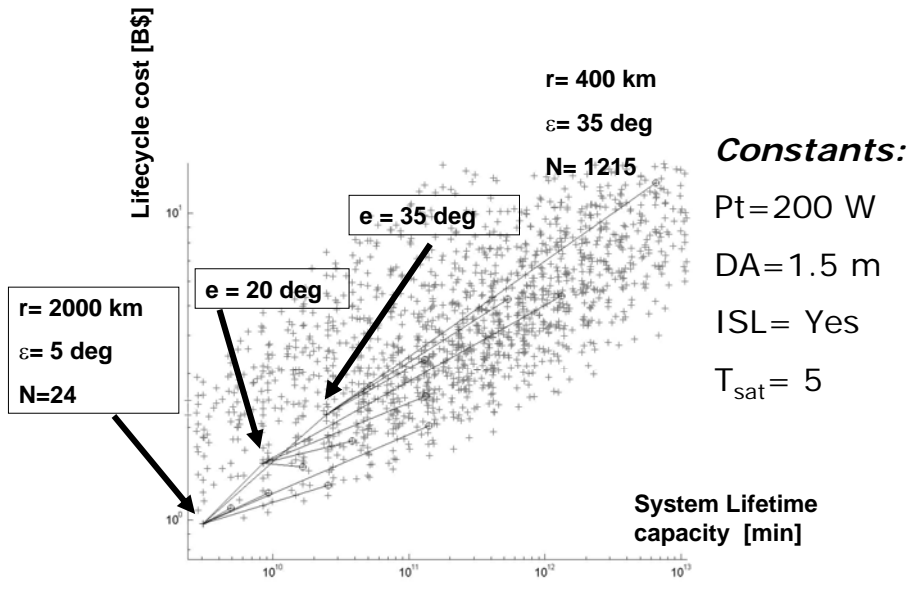
Trade Space



Path example



Tree example



Economies of Scale: Concept

- A possible characteristic of cost function
- Concept similar to returns to scale, except
 - * ratio of 'X_i' not constant
 - * refers to costs (economies) not "returns"
 - * Not universal (as RTS) but depends on local costs
- Economies of scale exist if costs increase slower than product

$$\text{Total cost} = C(Y) = Y^\alpha \quad \alpha < 1.0$$

Economies of Scale: Specific Cases

- If Cobb-Douglas, linear input costs, Increasing RTS ($r = \sum a_i > 1.0$)
=> Economies of scale

Optimality Conditions: $[a_1/X_1^*] / p_1 = [a_2/X_2^*] / p_2$

Thus: Inputs Cost is a function of X_1^*

Also: Output is a function of $[X_1^*]^r$

So: X_1^* is a function of $Y^{1/r}$

Finally: $C(Y) = \text{function of } [Y^{1/r}]$

Economies of Scale: General Case

Not necessarily true in general that Returns to scale => Economies of Scale

Increasing costs may counteract advantages of returns to scale

See example!!

$$c(Y) = c(\underline{X}^*) = (2^{-0.05})Y^{1.05}$$

Contrarily, if unit prices decrease with volume (quite common) we can have Economies of Scale, without Returns to Scale

Marginal Analysis: Summary

- **Assumptions --**
 - Convex Feasible Region
 - Unconstrained
- **Optimality Criteria**
 - * MC/MP same for all inputs
- **Expansion path -- Locus of Optimal Design**
- **Cost function -- cost along Expansion Path**
- **Economies of scale (vs Returns to Scale)**
 - Exist if Cost/Unit decreases with volume