Lattice Model of System Evolution

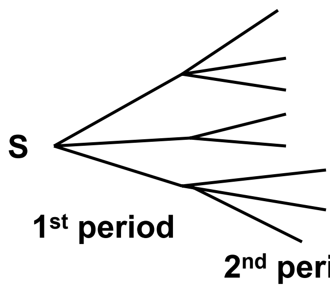
Richard de Neufville
Professor of Engineering Systems
and of
Civil and Environmental Engineering
MIT

Outline

- "Curse of Dimensionality"
- Binomial Lattice Model for Outcomes
  - Linear in Logarithms
  - Binomial lattice.xls
- Binomial Lattice Model for Probabilities
  - Normal distribution in logarithms
- Fitting to a known distribution
  - From average, standard deviation solve for u, d, p
- Underlying assumptions

System Evolution

- Think of how a system can evolve over time
  - It starts at State S
  - Over 1st period, it evolves into "i" states S_{1i}
  - In 2nd period, each S_{1i} evolves into more states…



Etc, etc...

"Curse of Dimensionality"

- Consider a situation where each state:
  - Evolves into only 2 new states…
  - Over only 24 periods (monthly over 2 years)
  - How many states at end?

ANSWER: 2, 4, 8 => 2^N = 2^{24} ~ 17 MILLION!!!

This approach swamps computational power

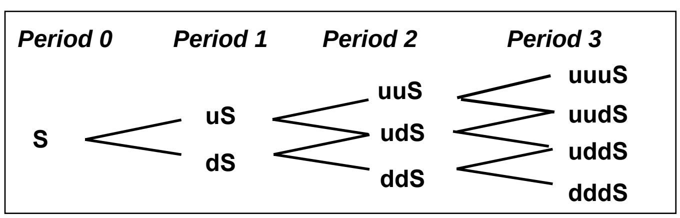
Binomial Lattice Model – 1st Stage

- **Assumes**
  - Evolution process is same over time (stationary)
  - Each state leads to only 2 others over a period
  - Later state is a multiple of earlier state
  - S => u S and d S (by convention, up> down)
- **For one period:**

What happens over 2nd , more periods?

Engineering Systems Analysis for Design
Massachusetts Institute of Technology
Richard de Neufville ©
Lattice Model Slide 5 of 32

Binomial Lattice: Several periods



- **States coincide**
  - path “up then down” => d(uS) = udS
  - same as “down then up” => u(dS) = udS
- **States increase linearly (1,2, 3, 4 => N not exponentially (1, 2, 4, 8…) = 2^N**
  - After 24 months: 25 states, not 17 million

Engineering Systems Analysis for Design
Massachusetts Institute of Technology
Richard de Neufville ©
Lattice Model Slide 6 of 32

Main Advantage of Binomial Model

- Eliminates "Curse of Dimensionality"
- Thus enables detailed analysis
  - Example: A binomial model of evolution every day of 2 years would only lead to 730 states, compared to ~17 million states resulting from 'decision tree' model of monthly evolution
- The jargon phrase is that Binomial is a <u>recombinatiorial</u> model…

Non-negativity of Binomial Model

- The Binomial Model does not allow shift from positive to negative values: lowest value (d^n S) is always positive
- This is realistic – indeed needed -- in many situations:
  - Value of an Asset; Demand for a Product; etc.
- Is non-negativity always realistic?
- NO! Contrary to some assumptions
  - Example: company profits! Easily negative

Path Independence: Implicit Assumption

Pay Attention – Important point often missed!

- Model Implicitly assumes "Path Independence"
  - Since all paths to a state have same result
  - Then value at any state is independent of path
  - In practice, this means nothing fundamental happens to the system (no new plant built, no R&D , etc)

When is "Path Independence" OK?

- Generally for Financial Options. Why?
  - Random process, no memory….

- Often not for Engineering Systems. Why?
  - If demand first rises, system managers may expand system, and have extra capacity when demand drops.
  - If demand drops then rises, they won't have extra capacity and their situation will differ
  - The process then depends on path!

Easy to develop in Spreadsheet

- Easy to construct by filling in formulas
- Class reference: Binomial lattice.xls
  - Allows you to play with numbers, try it
- Example for: S = 100; u = 1.2 ; d = 0.9

OUTCOME LATTICE

100.00	120.00	144.00	172.80	207.36	248.83	298.60
	90.00	108.00	129.60	155.52	186.62	223.95
		81.00	97.20	116.64	139.97	167.96
			72.90	87.48	104.98	125.97
				65.61	78.73	94.48
					59.05	70.86
						53.14

Relationship between States

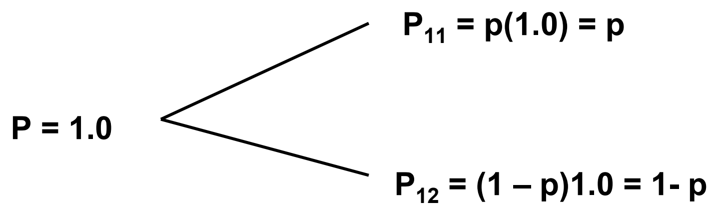
- The relative value between a lower and the next higher is constant = u /d

  S => uS and dS ; Ratio of uS /dS = u/d
- Thus results for 6th period, u/d = 1.2/.9 = 1.33

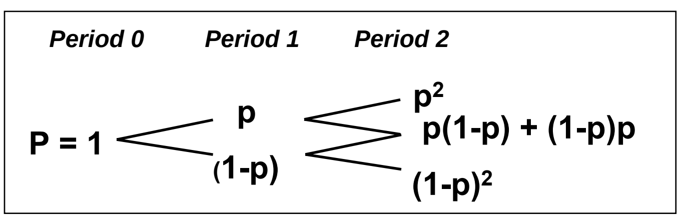
	Step	(u/d)exp[step]	0utcome/lowest
298.60	6	5.62	5.62
223.95	5	4.21	4.21
167.96	4	3.16	3.16
125.97	3	2.37	2.37
94.48	2	1.78	1.78
70.86	1	1.33	1.33
53.14	0	1.00	1.00

Application to Probabilities

- Binomial model can be applied to evolution of probabilities
- Since Sum of Probabilities = 1.0
  - Branches have probabilities: p ; (1- p)



Important Difference for Probabilities



- A major difference in calculation of states:
  - Values are not “path independent”
  - Probabilities = Sum of probabilities of all paths to get to state

Spreadsheet for Probabilities

- Class reference: Binomial lattice.xls
- Example for: p = 0.5 ; (1 –p) = 0.5
- => Normal distribution for many periods

PROBABILITY LATTICE

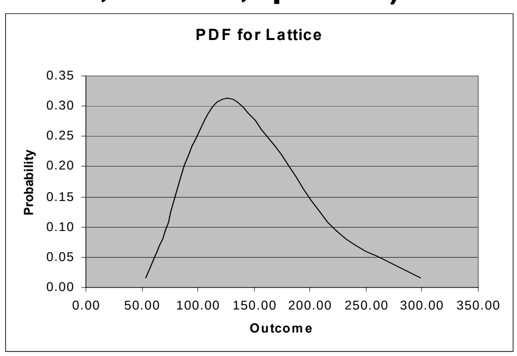
1.00	0.50	0.25	0.13	0.06	0.03	0.02
	0.50	0.50	0.38	0.25	0.16	0.09
		0.25	0.38	0.38	0.31	0.23
			0.13	0.25	0.31	0.31
				0.06	0.16	0.23
					0.03	0.09
						0.02

Outcomes and Probabilities together

- Applying Probability Model to Outcome Model leads to Probability Distribution on Outcomes
- In this case (u = 1.2 ; d = 0.9; p = 0.5):

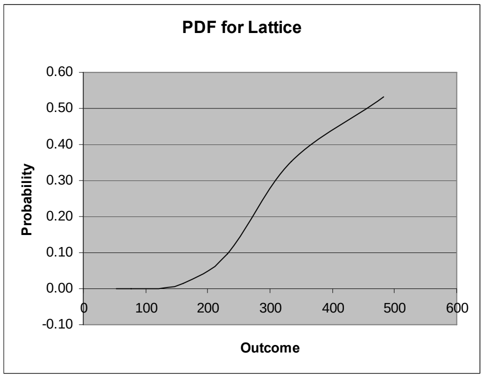
AXES

Outcome	Prob
298.60	0.02
223.95	0.09
167.96	0.23
125.97	0.31
94.48	0.23
70.86	0.09
53.14	0.02



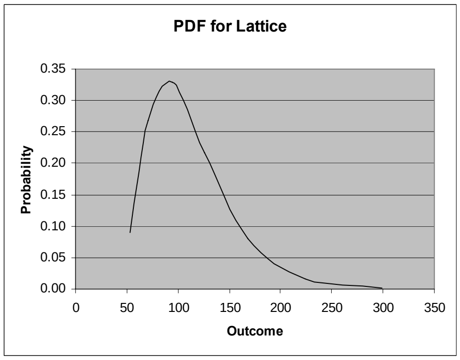
Many PDFs are possible

- For example, we can get “triangular right” with u = 1.3 ; d = 0.9 ; p = 0.9



Many PDFs are possible

- … or a ”skewed left”
  with u = 1.2 ; d = 0.9 ; p = 0.33



Let's try it

- An interlude with Binomial lattice.xls

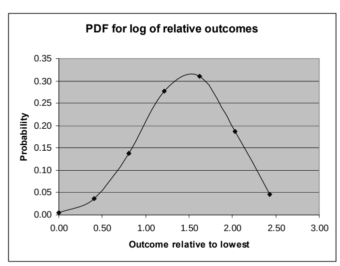
Calibration of Binomial Model

- Examples show that Binomial can model many PDF
- Question is: Given actual PDF, what is Binomial?
- Note that: Binomial pdf of outcomes is lognormal: "natural logs of outcomes are normally distributed"

For: u = 1.2 ; d = 0.8 ; p = 0.6

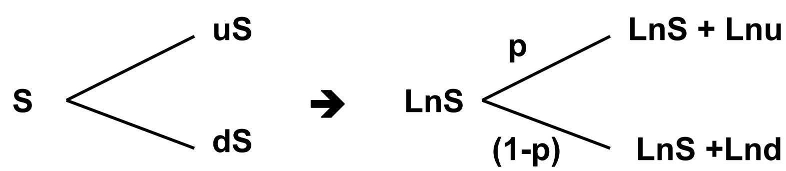
Note linearity of ln(outcomes):

outcome	Ln[(out/low)/LN(low)	**LN(outcome/lowest))**
298.60	6	2.43
199.07	5	2.03
132.71	4	1.62
88.47	3	**1.22**
58.98	2	0.81
39.32	1	0.41
26.21	0	0.00



Data from Actual PDF

- Two Elements of Observational (or Assumed) Data
  - Variance of PDF = σ^2 (= square of standard deviation)
  - Average, generally assumed to be growing at some rate, v, per period: $S_T = S e^{vT}$
  - Rate depends on length of period: 12%/year = 1%/month etc
  - Both v and σ are expressed in terms of <u>percentages!</u>
- Calibrate this to Ln (outcomes)



Two Conditions to be met

- Average increase over period:

  $v\Delta T = p \, Lnu + (1 - p) \, Lnd$

- Variance of distribution

  $\sigma^2 \Delta T = p \, (Lnu)^2 + (1- p) \, (Lnd)^2 - [p(Lnu) + (1-p)Lnd]^2$

  = Sum of weighted squares of observations

  -- [(average) squared]

- This has 2 equations and 3 unknowns (u, d, p)
  - Set: Lnu = - Lnd equivalent to u = 1/d

Solution for u ; d ; p

The previous equations can be solved, with a lot of "plug and chug" to get

$u = e \exp(\sigma \sqrt{\Delta t})$

$d = e \exp(-\sigma \sqrt{\Delta t})$

$p = 0.5 + 0.5 (v/\sigma) \sqrt{\Delta t}$

The calculated values can be used directly

Example Solution for u ; d ; p

- Assume that S = 2500 (e.g., $/ton of Cu Fine)
  v = 5% σ = 250 = 10% Δt = 1 year
- Then

$u = e \exp(\sigma \sqrt{\Delta t}) = e \exp(0.1) = 1.1052$

$d = e \exp(-\sigma \sqrt{\Delta t}) = 0.9048 = (1/u)$

$p = 0.5 + 0.5 (v/\sigma) \sqrt{\Delta t} = 0.75$

Note: everything varies with Δt

Assumption re Determining u, d, p

- The assumption behind these calculations is that actual PDF has a random (Gaussian, Normal) aspect to it

- Why? Or when is this reasonable?
- A 2-phase argument, first:
  - Project risks can be avoided by diversification
  - Thus only looks at market risk
- Second, that Markets are:
  - efficient, have "full information", and no bias
  - Thus error is random or "white noise"
- Thus random variations is usual assumption
  - See later discussion of GBM, Ito process…

Summary

- Lattice Model similar to a Decision Tree, but…
  - Nodes coincide
  - Problem size is linear in number of periods
  - Values at nodes defined by State of System
  - Thus "path independent" values

- Lattice Analysis widely applicable
  - With actual probability distributions
  - Accuracy depends on number of periods -- can be very detailed and accurate

- Reproduces uncertainty over time to simulate actual sequence of possibilities

APPENDIX

**ESTIMATING
LATTICE PARAMETERS
FROM ENGINEERING JUDGEMENTS**

Baseline Estimation Procedure

- When dealing with observations on a variable over time (for example, the price of a stock), the lattice parameters v and σ can easily be derived statistically.
- Keep in mind that we have a multiplicative function and exponential growth, e^{vt}
  - v, the average exponential growth, is the best fit of LN(data) against time
  - σ, the standard deviation, is defined by the differences between the observations and average growth

Issue with Engineering Data

- In Design, we may not have historical data from which we can derive v and σ
- We may have forecasts or estimates of future states, such as demand for a product
- For example, our estimate might be that demand would grow 20% +/- 15% in 5 years

How do we deal with this?

Dealing with Engineering Data (1)

- First, keep in mind that v and σ are <u>yearly</u> rates. If you any other period, you must adjust accordingly.
- Given 20% growth over 5 years, v ~ 4% [Strictly, the rate is lower, since we are considering exponential growth. However, the accuracy implied in a 20% growth rate does not justify precision beyond 1st decimal place]

Dealing with Engineering Data (2)

- σ can be estimated in a variety of ways
- Reasoning that uncertainty grows regularly over time, then +/- 15% over five years comes to +/- 3% in one year
- With 2 observations, a statistical estimate for σ is somewhat speculative. Within the accuracy of this process, however, the assumptions in the forecast imply $\sigma \sim 3\%$

Estimates of p

- The preceding estimates of v~ 4% and σ~3% would seem to present a problem…
- Inserting these values in slide 23, using Δ t =1
  $p = 0.5 + 0.5 (v/\sigma) \sqrt{\Delta t}$ => p > 1.0 !!!

This is impossible. What to do?

- Solution: scale down to shorter time, such as 3 months [Δ t = ¼], where v and σ also scale down. With this value we get:
  $p = 0.5 + 0.5 (4/3) (1/2) = 5/6 \sim 0.83$