Constrained Optimization – Part 1

- **Objective of Presentation: To introduce Lagrangean as a basic conceptual method used to optimize design in real situations**

- **Essential Reality: In practical situations, the designers are constrained or limited by**
  - **physical realities**
  - **design standards**
  - **laws and regulations, etc.**

Outline

- **Unconstrained Optimization (Review)**

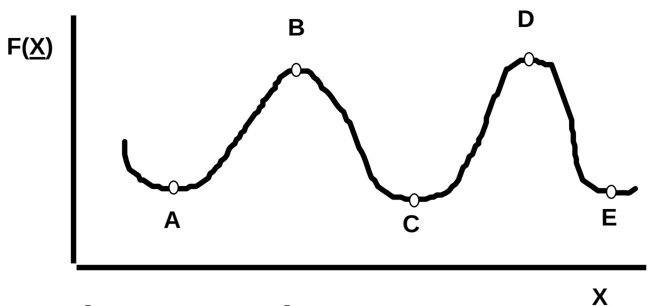
- **Constrained Optimization – Lagrangeans**
  - **Approach**
  - **Lagrangeans as Equality constraints**

- **Interpretation of Lagrangeans as "Shadow prices"**

Unconstrained Optimization: Definitions

- Optimization => Maximum of desired quantity,
  or => Minimum of undesired quantity
- Objective Function = Formula to be optimized
  = Z(X)
- Decision Variables = Variables about which we can make decisions
  = X = $(X_1 \ldots . X_n)$

Unconstrained Optimization: Graph



- B and D are maxima
- A, C and E are minima

Unconstrained Optimization: Conditions

- **By calculus: if F(X) continuous, analytic**
- **Primary conditions for maxima and minima:**

  $\partial F(X) / \partial X_i = 0 \quad \forall_i$

  (symbol means: "for all i")
- **Secondary conditions:**

  $\partial^2 F(\underline{X}) / \partial X_i^2 < 0$ = > Max (B,D)

  $\partial^2 F(\underline{X}) / \partial X_i^2 > 0$ = > Min (A,C,E)

  These define whether point of no change in Z is a maximum or a minimum

Unconstrained Optimization: Example

- **Example: Housing insulation**

  Total Cost = Fuel cost + Insulation cost

  x = Thickness of insulation

  $F(x) = K_1 / x + K_2 x$

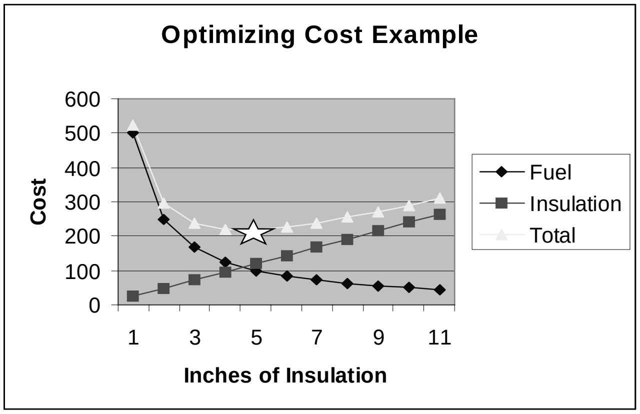
  Primary condition: $\partial F(x) / \partial x = 0 = -K_1 / x^2 + K_2$

  $\Rightarrow x^* = \{K_1 / K_2\}^{1/2}$

  (starred quantities are optimal)

Unconstrained Optimization: Graph of Solution to Example

- **If: $K_1 = 500$; $K_2 = 24$ Then: $X^* = 4.56$**



Constrained Optimization: General

- **"Constrained Optimization" involves the optimization of a process subject to constraints**

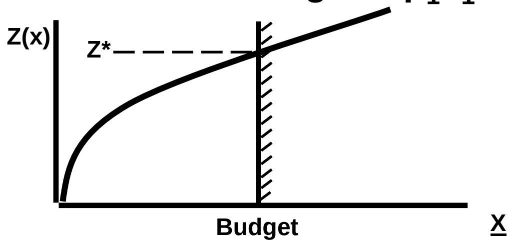
- **Constraints have two basic types**
  - **<u>Equality</u> Constraints -- some factors have to equal constraints**
  - **<u>Inequality</u> Constraints -- some factors have to be less less or greater than the constraints (these are "upper" and "lower" bounds)**

Constrained Optimization: General Approach

- To solve situations of increasing complexity, (for example, those with equality, inequality constraints) …
- Transform more difficult situation into one we know how to deal with
- Note: this process introduces new variables!
- Thus, transform
  - "constrained" optimization to "unconstrained" optimization

Equality Constraints: Example

- Example: Best use of budget
- Maximize: Output = $Z(\underline{X}) = a_0 x_1^{a1} x_2^{a2}$
- Subject to (s.t.):
  Total costs = Budget = $p_1x_1 + p_2x_2$



Note: $\partial Z(\underline{X}) / \partial \underline{X} \neq 0$ at optimum

Lagrangean Method: Approach

- Transforms equality constraints into unconstrained problem
- Start with:

  Opt: $F(\underline{x})$
  s.t.: $g_j(\underline{x}) = b_j \quad => \quad g_j(\underline{x}) - b_j = 0$
- Get to: $L = F(\underline{x}) - \Sigma_j \lambda_j[g_j(\underline{x}) - b_j]$

  λ_j = Lagrangean multipliers (lambdas sub j) -- are unknown quantities for which we must solve

Note: $[g_j(\underline{x}) - b_j] = 0$ by definition, thus
optimum for $F(\underline{x})$ = optimum for L

Lagrangean: Optimality Conditions

- Since the new formulation is a single equation, we can use formulas for unconstrained optimization.
- We set partial derivatives equal to zero for all unknowns, the X and the λ
- Thus, to optimize L:

  $\partial L / \partial x_i = 0 \qquad \forall_I$
  $\partial L / \partial \lambda_j = 0 \qquad \forall_J$

Lagrangean: Example Formulation

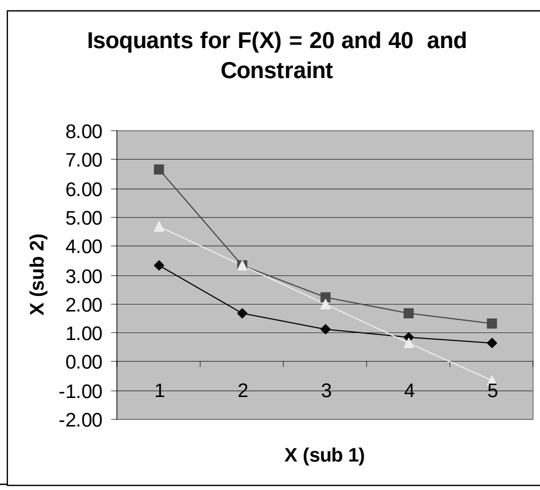
- **Problem:**
  Opt: $F(\underline{x}) = 6x_1x_2$
  s.t.: $g(\underline{x}) = 3x_1 + 4x_2 = 18$

- **Lagrangean:**
  $L = 6x_1x_2 - \lambda(3x_1 + 4x_2 - 18)$

Optimality Conditions:

$$\partial L / \partial x_1 = 6x_2 - 3\lambda = 0$$
$$\partial L / \partial x_2 = 6x_1 - 4\lambda = 0$$
$$\partial L / \partial \lambda_j = 3x_1 + 4x_2 - 18 = 0$$

Lagrangean: Graph for Example



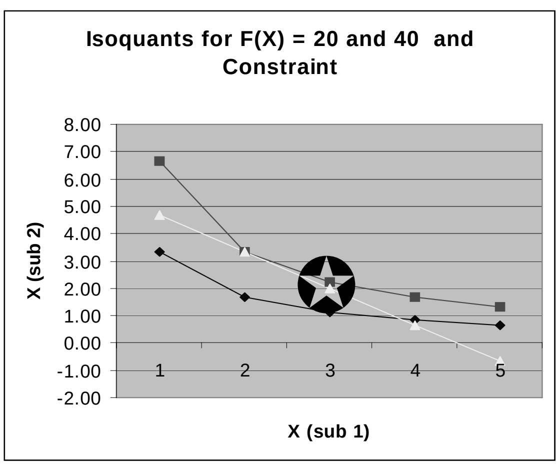
Lagrangean: Example Solution

- **Solving as unconstrained problem:**

$$\partial L / \partial x_1 = \quad 6x_2 - 3\lambda = 0$$
$$\partial L / \partial x_2 = \quad 6x_1 - 4\lambda = 0$$
$$\partial L / \partial \lambda_i \; = \quad 3x_1 + 4x_2 - 18 = 0$$

- **so that:** $\lambda = 2x_2 = 1.5x_1$ **(first 2 equations)**
  => $x_2 = 0.75x_1$
  => $3x_1 + 3x_1 - 18 = 0$ **(3rd equation)**
- $x_1^* = 18/6 = 3 \quad x_2^* = 18/8 = 2.25 \quad \lambda^* = 4.5$
- $F(x)^* = 40.5$

Lagrangean: Graph for Solution



Shadow Price

- Shadow Price = Rate of change of objective function per unit change of constraint
  $= \partial F(\underline{x}) / \partial b_j = \lambda_j$

- It is extremely important for system design

- It defines value of changing constraints, and indicates if worthwhile to change them
  - Should we buy more resources?
  - Should we change environmental constraints?

Lagrangean Multiplier is a Shadow Price

- The Lagrangean multiplier is interpreted as the shadow price on constraint

- $SP_j = \partial F(\underline{x})^* / \partial b_j = \partial L^* / \partial b_j$
  $= \partial \{F(\underline{x}) - \Sigma_j \lambda_j [g_j(\underline{x}) - b_j]\} / \partial b_j$
  $= \lambda_j$
  - Naturally, this is an instantaneous rate

Lagrangean = Shadow Prices Example

- Let's see how this works in example, by changing constraint by 0.1 units:
  Opt: $F(\underline{x}) = 6x_1x_2$
  s.t.: $g(\underline{x}) = 3x_1 + 4x_2 = 18.1$
- The optimum values of the variables are
  $x_1^* = (18.1)/6 \quad x_2^* = (18.1)/8 \qquad \lambda^* = 4.5$
- Thus $F(x)^* = 6(18.1/6)(18.1/8) = 40.95$

$$\Delta F(x) = 40.95 - 40.5 = 0.45 = \lambda^* (0.1)$$

Generalization

- In general constraints are "inequalities":
  - Upper bounds: $g_j(\underline{x}) \leq b_j$
  - Lower bounds: $g_j(\underline{x}) \geq b_j$
- At optimum, some constraints will limit solution (they are "binding") others not
  - Example: airline bags: weight ≤ 40kg ; sum of dimensions ≤ 2.5m. Your bag might be limited by weight, not by size.
- Shadow prices
  ≥ 0 for all "binding" constraints
  = 0 for all others

Design Implications

- Expanding range of design variables (<u>x</u>), increases freedom to improve design, thus adds value
- This is called "relaxing" the constraints
  - Increasing upper bounds
  - Decreasing lower bounds
- As any constraint relaxed, it may no longer be "binding" , and others can become so
- SHADOW PRICES DEPEND ON OTHER CONSTRAINTS, "PROBLEM DEPENDENT"

Take-aways

- Relaxing design constraints adds value (in terms of better performance, F(x))
- This value is the "shadow price" of that constraint
- Knowing this can be very important for designers, shows way to improve quickly
- NOTE: Value to design has no direct connection to cost of constraint, not a "price in ordinary terms