Arbitrage Enforced Valuation

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Outline

- Replicating Portfolio – key concept
- Motivating Example ➔ Value Independent of Objective probabilities!!!
- Arbitrage Enforced Pricing
- Application to Binomial Lattice Analysis
- Risk Neutral "probabilities"

Definition: Replicating Portfolio

- A "replicating portfolio" is..
- A set of assets (a portfolio) that has same payoffs – replicates – payoffs of option

- Example for a "call" option
  - If asset value goes up, exercise option and option profit = portfolio profit
  - If asset value down, do not exercise option and option value = 0 = portfolio value

- Note: Replicating Option is not obvious, … must be constructed carefully

Use of Replicating Portfolio

- Why is a Replicating Portfolio Useful?

- … Because it may be easier to value portfolio than option

- Since by construction the portfolio exactly replicates payoffs of option
- Thus: value of portfolio ≡ value of option
- So we can value option as sum of values of replicating portfolio

  … as example will show

Basis for Replicating Portfolio -- Call

Think about what a call option provides…

- **It enables owner to get possible increased value of asset**
  - **If exercised, call option results in asset ownership**
- **However, it provides this benefit without much money! Payment for asset delayed until option is exercised**
  - **Ability to delay payment is equivalent to a loan**
- **Therefore: A Call option is like buying asset with borrowed money**

Basis for Replicating Portfolio -- Put

Argument is similar for a put…

- **Put enables owner to avoid possible loss in value of asset**
  - **If exercised, put option results in sale of asset**
- **However, it provides this benefit without early commitment! Delivery of asset delayed until option is exercised**
  - **Ability to delay delivery is equivalent to a loan**
- **Therefore: A Put option is like getting cash (i.e., selling asset) with borrowed asset**

Example will illustrate

- Explanation for replicating portfolio (e.g., call as "buying asset with a loan") … is not obvious
- Example will help, but you will need to think about this to develop intuition
- Bear with the development!

Motivating Example

- Valuation of an example simple option has fundamentally important lessons
- Key idea is possibility of replicating option payoffs using a portfolio of other assets
  - Since option and portfolio have same payoffs,
  - The value of option = value of portfolio
- Surprisingly, <u>when replication possible</u>:

value of option does NOT
depend on objective probability of payoffs!

Motivating Example – Generality

- The following example illustrates how a replicating portfolio works in general

- The example makes a specific assumption about how the value of the asset moves…
- The principle used to replicate the option does not depend on this assumption, it can be applied to any assumption
- Once you make assumption about how the asset moves, it is possible to create a replicating portfolio

A Simple 1-Period Option

- Asset has a Current price, S_0 = $100

- Price at end of period either
  S_{DOWN} = $80 or S_{UP} = $125

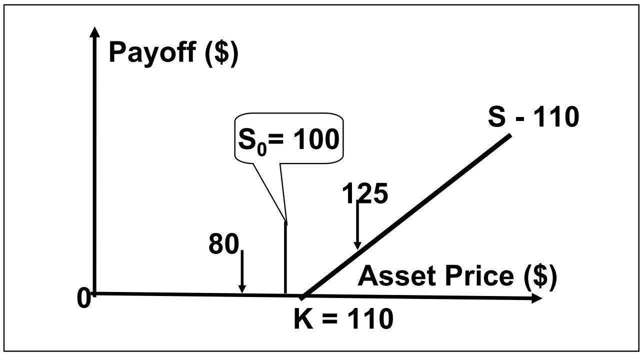
- One-period call option to buy asset at Strike price, K = $110

- What is the value of this option?

- More precisely, what is the maximum price, C, that we should pay for this option?

Graphically…

- Call Option on S, S currently worth 100 = S_0
- strike price = K = 110
- possible values of S: S_{DOWN} = 80 ; S_{UP} = 125



Call Option Cost and Payoffs

- Fair Cost of Option, C, is its value. This is what we want to determine

- If at end of period
- asset price > strike price: option payoff = S – K
- asset price < strike price: option payoff = 0

	Start	End	End
Asset Price	100	80	125
Buy Call Strike = 110	- C	0	(125 - 110) = 15

Replicating Portfolio Cost and Payoffs

- Replicating Portfolio consists of:
  Asset bought at beginning of period
  Financed in part by borrowed money

- Amounts of Asset bought and money borrowed arranged so that payoffs equal those of option
- Specifically, need to have asset and loan payment to net out as follows:
  - If S > K, want net = positive return
  - If S < K, want net = 0

Note: This is first crucial point of arrangement!

Creating the replicating portfolio

- This description designed to show what is going on – in practice, short-cut procedure is used

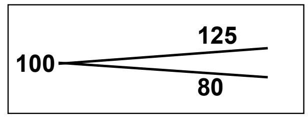
- Recognize that (net value of portfolio)
  = (asset value – loan repayment)
- To arrange that (net value of portfolio) = 0…
- We set: (loan repayment) = S_{DOWN} = 80

- Note: (loan repayment)
  = (amount borrowed) + (interest for period)
  = (amount borrowed) (1 + r)
- (Amount borrowed) = 80 / (1+ r)

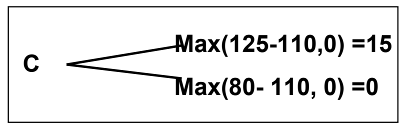
Graphically…

- The situation has 3 elements

Value of Asset is
up or down



Value of call option
Is up or down



Value of loan rises
by r over period
Need to repay R = 1 + r

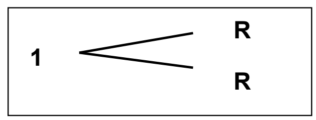


Table of Portfolio Cost and Payoffs

	Start	End	End
Asset price	100	80	125
Buy Asset	-100	80	125
Borrow Money	80/(1+r)	- 80	- 80
Net	-100 + 80/(1+r)	0	45

Observe: End net of portfolio is like payoff of option

Comparing Costs and Payoffs of Option and Replicating Portfolio

- If S < K, both payoffs automatically = 0 by design
- If S > K, call payoff is a multiple of portfolio payoff (in this case, ratio is 1:3)
- Thus: payoffs of call = payoffs of (1/3) portfolio
- Also: Net cost of portfolio = - [asset cost – loan]

Period	Start	End	End
Asset Price	100	80	125
Buy Call	- C	0	(125-110) = 15
Buy Asset And Borrow	-100 + 80/(1 + r)	0	45

Value of Option (1)

- Value of Option = (1/3) (Value of Portfolio)
- C = (1/3)[-100 + 80/ (1 + r)]

… calculated at appropriate r -- What is that?

Period	Start	End	End
Asset Price	100	80	125
Buy 3 Calls	- 3C	0	45
Buy Asset And Borrow	-100 + 80/(1 + r)	0	45

Implications of: Option = Portfolio

- Crucial observation:
- Seller of option can counter- balance this with a portfolio of equal value, and thus arrange it so cannot lose!
- Such a no-risk situation is known as ARBITRAGE
- Since arbitrage has no risk, appropriate DISCOUNT RATE = Rf = RISK FREE RATE!
- This is second crucial point of arrangement

Value of Option (2)

- The appropriate value of option is thus
- assuming Rf = 10% (for easy calculation)
- C = (1/3)[-100 + 80/ (1 + Rf)] = $ 9.09

Period	Start	End	End
Asset Price	100	80	125
Buy 3 Calls	- 3C	0	45
Buy Asset And Borrow	-100 + 80/(1 + r) = 9.09	0	45

Value independent of actual probability!

- Nowhere in calculation of the option value is there any statement about actual probability that high or low payoffs (125 or 80) occur
- In situations as described,
  actual probabilities do not matter!
- Very surprising, since options deal with uncertainties
- A remarkable, counter-intuitive result
- What matters is the RANGE of payoffs

No knowledge or need for PDF

- Recognize that all the only thing we needed to know about the asset was the possible set of outcomes at end of stage
- The "asset" is like a black box – we know
  - what comes in (in this case S = 100)
  - What comes out (in this case, 80 or 125)
- Nowhere do we know anything about PDF
- With arbitrage-enforced pricing, we are not in Expected Value world (in terms of frequency)

Arbitrage-Enforced Pricing -- Concept

- Possibility of setting up a replicating portfolio to balance option absolutely defines value (and thus market price) of option
- Replicating portfolio permits market pressures to drive the price of option to a specific value
- This is known as "Arbitrage-Enforced Pricing"
- THIS IS CRUCIAL INSIGHT!!!
- It underlies all of options analysis in <u>finance</u> (note implied restriction)

Arbitrage-Enforced Pricing -- Mechanism

- How does "Arbitrage - Enforced Pricing" work?
- If you are willing to buy option for C* > C, the price defined by portfolio using risk-free rate
- then someone could sell you options and be sure to make money -- until you lower price to C
- Conversely, if you would sell option for C_* < C, then someone could buy them and make money until option price = C
- Thus: C is price that must prevail

Arbitrage-Enforced Pricing -- Assumptions

- When does "Arbitrage-Enforced Pricing" work?

- Note key assumptions:
- Ability to create a "replicating portfolio"
  - This is possible for financially traded assets
  - May not be possible for technical systems (for example, for a call on extra capacity, or use of spare tire for car)
- Ability to conduct arbitrage by buying or selling options and replicating portfolios
  - There may be no market for option or portfolio, so price of option is not meaningful, and market pressure cannot be exercised
- If Assumptions not met, concept dubious

Arbitrage-Enforced Pricing -- Applicability

- For options on traded assets (stocks, foreign exchange, fuel, etc.), it is fair to assume that conditions for arbitrage-enforced pricing exist
- Arbitrage-enforced pricing thus a fundamental part of traditional "options analysis"

- For real options, "on" and "in" technical systems, the necessary assumptions may not hold
- It is an open issue whether and when arbitrage-enforced pricing should be used

- In any case: You need to know about it!

Basis for Options analysis

- The valuation of this very simple option has fundamentally important lessons

- Surprisingly, when replication possible:
  value of option does NOT
  depend on probability of payoffs!
  - Contrary to intuition associated with probabilistic nature of process
  - This surprising insight is basis for options analysis

Application to Binomial Lattice

- How does arbitrage-enforced pricing apply to the binomial lattice?

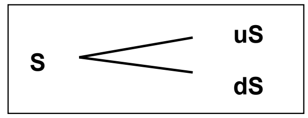
- It replaces actual binomial probabilities
  - (as set by growth rate, v, and standard deviation, σ)
- by relative weights derived from replicating portfolio

- These relative weights reflect the proportion ratio of asset and loan (as in example) – but look like probabilities: they are called the risk-neutral "probabilities"

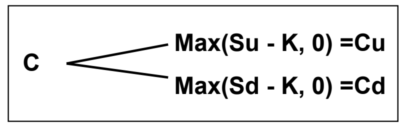
Single Period Binomial Model Set-up

- Apply to generalized form of example

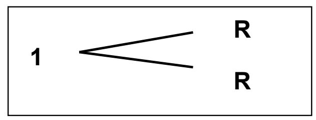
Value of Asset is up or down



Value of call option Is up or down



Value of loan rises by Rf (no risk) to R = 1 + Rf (for 1 year)



Single Period Binomial Model Solution

- The issue is to find what proportion of asset and loan to have to establish replicating portfolio (This time we replicate exactly)
- Set: asset share = "x" loan share = "y"
- then solve:
  xuS + yR = Cu and xdS + yR = Cd

=> x = (Cu - Cd) / S(u - d) from Cu - Cd

=> y = (1/R) [uCd - dCu] / (u - d) by substitution

Single Period Binomial Model Solution

- Now to find out the value of the option
- Portfolio Value = Option Price

$$= xS + y(1)$$

$$= (Cu - Cd) / (u\text{-}d) + (uCd - dCu) / R(u\text{-}d)$$

$$= \{(R - d)Cu + (u - R)Cd\} / \{R(u\text{-}d)\}$$

Application to Example

For Example Problem:
R = 1 + Rf = 1.1 (Rf assumed = 10% for simplicity)
Cu = value of option in up state = 15
Cd = value of option in down state = 0
u = ratio of up movement of S = 1.25
d = ratio of down movement of S = 0.8

Portfolio Value = Option Price
= [(R - d)Cu + (u - R)Cd] / R(u-d)
= [(1.1- 0.8) (15) + (1.25 - 1.1)(0)] / 1.1(1.25 - 0.8)
= [0.3(15)] / 1.1(.45) = 10 / 1.1
= 9.09 as before

Reformulation of Binomial Formulation

Option Price = {(R - d)Cu + (u - R)Cd} / {R(u-d)}

- We simplify writing of formula by substituting a single variable for a complex one:

  "q" ≡ (R - d) / (u - d)

- Option Price = {(R - d)Cu + (u - R)Cd} / {R(u-d)}
  = (1/R) [{(R-d)/(u-d)}Cu + {(u-R)/(u-d)}Cd]
  = (1/R) [q Cu + (1- q) Cd)]

Option Value is weighted average of q, (1 – q)

q factor = risk-neutral "probability"

- Option Price = (1/R) [q Cu + (1- q) Cd)]
- This leads to an extraordinary interpretation! Value of option = "expected value" with binomial probabilities q and (1 - q)
- These called: "risk- neutral probabilities"
- Yet "q" defined by spread: q ≡ (R - d) / (u - d) actual probabilities do not enter into calculation!

$$C \begin{cases} \xrightarrow{q} \text{Max(Su - K, 0) = Cu} \\ \xrightarrow{(1-q)} \text{Max(Sd - K, 0) = Cd} \end{cases}$$

Binomial Procedure using q

- “Arbitrage-enforced” pricing of options in binomial lattice proceeds as with “decision analysis” based valuation covered earlier

- Difference is that probabilities are no longer (p, 1 - p) but (q, 1 - q)

- From the perspective of calculation, (q, 1 - q) are exactly like probabilities
- However, never observed as frequencies, etc.
- Said to be “risk-neutral”, because derived from assumption of risk-free arbitrage

Summary

- Replicating Portfolio – key concept
  - A combination of asset and loan
  - Designed to give same outcomes as option
  - Leads to possibility of valuation of option

- Arbitrage Enforced Pricing
  - Option value determined by replicating portfolio
  - Provided that assumptions hold
  - “always” for traded assets;
  - Unclear for real options

- Risk Neutral “probabilities”
  - Represent Arbitrage-enforced valuation in lattice

Appendix

Another way to appreciate the Rationale for risk-free discount rate In arbitrage enforced pricing of options

(with thanks to Michael Hanowsky)

Thought experiment

- We started with a call option on an asset that could end up with prices of 125 or 80
  - The Strike Price, K = 110
  - The option outcomes would be 15 or 0
- Suppose now a put on this asset with
  - Strike Price, K = 95
  - The option outcomes would be 0 or 15
- Now think of owning both together: the outcome is +15 whatever the asset price

Interpretation of Thought Case

- Both the Call and the Put are equivalent to a combination of the asset and borrowed money…
- The question for estimating the value of the options depends on the value assigned to r for the borrowed money…
- Since we can construct risk-free outcomes (as on previous slide)

Risk-free discount rate is appropriate!