Arbitrage Enforced Valuation

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Outline

- Replicating Portfolio – key concept
- Motivating Example ➔ Value Independent of Objective probabilities!!
- Arbitrage Enforced Pricing
- Application to Binomial Lattice Analysis
- Risk Neutral “probabilities”
Definition: Replicating Portfolio

- A “replicating portfolio” is..
- A set of assets (a portfolio) that has same payoffs – replicates – payoffs of option

Example for a “call” option
- If asset value goes up, exercise option and option profit = portfolio profit
- If asset value down, do not exercise option and option value = 0 = portfolio value

Note: Replicating Option is not obvious, … must be constructed carefully

Use of Replicating Portfolio

- Why is a Replicating Portfolio Useful?
  - … Because it may be easier to value portfolio than option

  - Since by construction the portfolio exactly replicates payoffs of option
  - Thus: value of portfolio ≡ value of option
  - So we can value option as sum of values of replicating portfolio
    - … as example will show
Basis for Replicating Portfolio -- Call

Think about what a call option provides…

- It enables owner to get possible increased value of asset
  - If exercised, call option results in asset ownership
- However, it provides this benefit without much money! Payment for asset delayed until option is exercised
  - Ability to delay payment is equivalent to a loan
- Therefore: A Call option is like buying asset with borrowed money

Basis for Replicating Portfolio -- Put

Argument is similar for a put…

- Put enables owner to avoid possible loss in value of asset
  - If exercised, put option results in sale of asset
- However, it provides this benefit without early commitment! Delivery of asset delayed until option is exercised
  - Ability to delay delivery is equivalent to a loan
- Therefore: A Put option is like getting cash (i.e., selling asset) with borrowed asset
Example will illustrate

- Explanation for replicating portfolio (e.g., call as “buying asset with a loan”) ...
  is not obvious

- Example will help, but you will need to think about this to develop intuition

- Bear with the development!

Motivating Example

- Valuation of an example simple option has fundamentally important lessons

- Key idea is possibility of replicating option payoffs using a portfolio of other assets
  - Since option and portfolio have same payoffs,
  - The value of option = value of portfolio

- Surprisingly, when replication possible:
  value of option does NOT depend on objective probability of payoffs!
Motivating Example – Generality

- The following example illustrates how a replicating portfolio works in general

- The example makes a specific assumption about how the value of the asset moves...
- The principle used to replicate the option does not depend on this assumption, it can be applied to any assumption
- Once you make assumption about how the asset moves, it is possible to create a replicating portfolio

A Simple 1-Period Option

- Asset has a Current price, $S_0 = $100

- Price at end of period either
  $S_{\text{DOWN}} = $80 \text{ or } S_{\text{UP}} = $125$

- One-period call option to buy asset at Strike price, $K = $110

- What is the value of this option?

- More precisely, what is the maximum price, $C$, that we should pay for this option?
Graphically...

- Call Option on S, S currently worth 100 = $S_0$
- strike price = $K = 110$
- possible values of S: $S_{DOWN} = 80; S_{UP} = 125$

\[
\begin{array}{c}
\text{Payoff ($)} \\
\text{Asset Price ($)} \\
S_0 = 100 \\
80 \\
125 \\
K = 110
\end{array}
\]

Fair Cost of Option, C, is its value. This is what we want to determine

- If at end of period
  - asset price > strike price: option payoff = $S - K$
  - asset price < strike price: option payoff = 0

<table>
<thead>
<tr>
<th>Asset Price</th>
<th>Start</th>
<th>End</th>
<th>End</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy Call</td>
<td>- C</td>
<td>0</td>
<td>(125 - 110) = 15</td>
</tr>
</tbody>
</table>
Replicating Portfolio Cost and Payoffs

- Replicating Portfolio consists of:
  - Asset bought at beginning of period
  - Financed in part by borrowed money

- Amounts of Asset bought and money borrowed arranged so that payoffs equal those of option
- Specifically, need to have asset and loan payment to net out as follows:
  - If \( S > K \), want net = positive return
  - If \( S < K \), want net = 0

Note: This is first crucial point of arrangement!

Creating the replicating portfolio

- This description designed to show what is going on – in practice, short-cut procedure is used

- Recognize that (net value of portfolio)
  \[ = \text{(asset value} - \text{loan repayment}) \]
- To arrange that (net value of portfolio) = 0…
- We set: (loan repayment) = \( S_{\text{DOWN}} = 80 \)

- Note: (loan repayment)
  \[ = (\text{amount borrowed}) + (\text{interest for period}) \]
  \[ = (\text{amount borrowed}) (1 + r) \]
- (Amount borrowed) = \( \frac{80}{(1 + r)} \)
The situation has 3 elements:

- Value of Asset is up or down
  
  ![Graph of Asset Value](image)
  
  Value of Asset:
  
  \[
  \text{Max}(125-110, 0) = 15 \\
  \text{Max}(80-110, 0) = 0
  \]

- Value of call option is up or down
  
  ![Graph of Call Option Value](image)
  
- Value of loan rises by \( r \) over period
  
  Need to repay \( R = 1 + r \)

Observe: End net of portfolio is like payoff of option

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Start</th>
<th>End</th>
<th>End</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset price</td>
<td>100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Buy Asset</td>
<td>-100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Borrow Money</td>
<td>( \frac{80}{1+r} )</td>
<td>-80</td>
<td>-80</td>
</tr>
<tr>
<td>Net</td>
<td>(-100 + \frac{80}{1+r})</td>
<td>0</td>
<td>45</td>
</tr>
</tbody>
</table>
Comparing Costs and Payoffs of Option and Replicating Portfolio

- If $S < K$, both payoffs automatically = 0 by design
- If $S > K$, call payoff is a multiple of portfolio payoff (in this case, ratio is 1:3)
- Thus: payoffs of call = payoffs of $(1/3)$ portfolio
- Also: Net cost of portfolio = - [asset cost – loan]

<table>
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<th>Period</th>
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<th>End</th>
<th>End</th>
</tr>
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<tr>
<td>Asset Price</td>
<td>100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Buy Call</td>
<td>- C</td>
<td>0</td>
<td>$(125-110) = 15$</td>
</tr>
<tr>
<td>Buy Asset</td>
<td>-100 + 80/(1 + r)</td>
<td>0</td>
<td>45</td>
</tr>
<tr>
<td>And Borrow</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Value of Option (1)

- Value of Option = $(1/3)$ (Value of Portfolio)
- $C = (1/3)[ -100 + 80/ (1 + r)]$
- ... calculated at appropriate r -- What is that?

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<tr>
<td>Asset Price</td>
<td>100</td>
<td>80</td>
<td>125</td>
</tr>
<tr>
<td>Buy 3 Calls</td>
<td>- 3C</td>
<td>0</td>
<td>45</td>
</tr>
<tr>
<td>Buy Asset And Borrow</td>
<td>-100 + 80/(1 + r)</td>
<td>0</td>
<td>45</td>
</tr>
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</table>
Implications of: Option = Portfolio

- Crucial observation:
  - Seller of option can counter-balance this with a portfolio of equal value, and thus arrange it so cannot lose!

- Such a no-risk situation is known as ARBITRAGE

- Since arbitrage has no risk, appropriate DISCOUNT RATE = Rf = RISK FREE RATE!

- This is second crucial point of arrangement

Value of Option (2)

- The appropriate value of option is thus
- assuming Rf = 10% (for easy calculation)
- \( C = \frac{1}{3}[-100 + \frac{80}{1 + Rf}] = $9.09 \)

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</tr>
<tr>
<td>Buy Asset</td>
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<td>0</td>
<td>45</td>
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Value independent of actual probability!

- Nowhere in calculation of the option value is there any statement about actual probability that high or low payoffs (125 or 80) occur

- In situations as described, actual probabilities do not matter!
- Very surprising, since options deal with uncertainties
- A remarkable, counter-intuitive result

- What matters is the RANGE of payoffs

No knowledge or need for PDF

- Recognize that all the only thing we needed to know about the asset was the possible set of outcomes at end of stage

- The “asset” is like a black box – we know
  - what comes in (in this case $S = 100$)
  - What comes out (in this case, 80 or 125)

- Nowhere do we know anything about PDF

- With arbitrage-enforced pricing, we are not in Expected Value world (in terms of frequency)
Arbitrage-Enforced Pricing -- Concept

- Possibility of setting up a replicating portfolio to balance option absolutely defines value (and thus market price) of option
- Replicating portfolio permits market pressures to drive the price of option to a specific value
- This is known as “Arbitrage-Enforced Pricing”

THIS IS CRUCIAL INSIGHT!!!
- It underlies all of options analysis in finance (note implied restriction)

Arbitrage-Enforced Pricing -- Mechanism

- How does “Arbitrage - Enforced Pricing” work?
- If you are willing to buy option for \( C^* > C \), the price defined by portfolio using risk-free rate
  then someone could sell you options and be sure to make money -- until you lower price to \( C \)
- Conversely, if you would sell option for \( C_* < C \), then someone could buy them and make money until option price = \( C \)
- Thus: \( C \) is price that must prevail
Arbitrage-Enforced Pricing -- Assumptions

- When does “Arbitrage-Enforced Pricing” work?

- Note key assumptions:
  - Ability to create a “replicating portfolio”
    - This is possible for financially traded assets
    - May not be possible for technical systems (for example, for a call on extra capacity, or use of spare tire for car)
  - Ability to conduct arbitrage by buying or selling options and replicating portfolios
    - There may be no market for option or portfolio, so price of option is not meaningful, and market pressure cannot be exercised
  - If Assumptions not met, concept dubious

Arbitrage-Enforced Pricing -- Applicability

- For options on traded assets (stocks, foreign exchange, fuel, etc.), it is fair to assume that conditions for arbitrage-enforced pricing exist
  - Arbitrage-enforced pricing thus a fundamental part of traditional “options analysis”

- For real options, “on” and “in” technical systems, the necessary assumptions may not hold
- It is an open issue whether and when arbitrage-enforced pricing should be used

- In any case: You need to know about it!
Basis for Options analysis

- The valuation of this very simple option has fundamentally important lessons

- Surprisingly, when replication possible:
  - value of option does NOT
  - depend on probability of payoffs!

  - Contrary to intuition associated with probabilistic nature of process
  - This surprising insight is basis for options analysis

Application to Binomial Lattice

- How does arbitrage-enforced pricing apply to the binomial lattice?

- It replaces actual binomial probabilities
  - (as set by growth rate, \( v \), and standard deviation, \( \sigma \))
- by relative weights derived from replicating portfolio

- These relative weights reflect the proportion ratio of asset and loan (as in example) – but look like probabilities: they are called the risk-neutral “probabilities”
Single Period Binomial Model Set-up

- Apply to generalized form of example

Value of Asset is up or down

Value of call option is up or down

Value of loan rises by Rf (no risk) to

\[ R = 1 + Rf \] (for 1 year)

\[
\begin{align*}
\text{S} & \quad \text{i} \quad \text{uS} \\
\text{dS} & \\
\text{C} & \quad \text{Max(Su - K, 0) = Cu} \\
& \quad \text{Max(Sd - K, 0) = Cd} \\
\text{1} & \quad \text{R} \\
\text{R} &
\end{align*}
\]

Single Period Binomial Model Solution

- The issue is to find what proportion of asset and loan to have to establish replicating portfolio (This time we replicate exactly)

- Set: asset share = “x” loan share = “y”

- then solve:

\[
\begin{align*}
x \cdot uS + y \cdot R &= Cu \\
xdS + y \cdot R &= Cd
\end{align*}
\]

\[
\Rightarrow x = \frac{(Cu - Cd)}{S \cdot (u - d)} \quad \text{from} \quad Cu - Cd
\]

\[
\Rightarrow y = \frac{(1/R) \cdot [uCd - dCu]}{(u - d)} \quad \text{by substitution}
\]
Single Period Binomial Model Solution

- Now to find out the value of the option

- Portfolio Value = Option Price

  \[ P = xS + y(1) = (Cu - Cd) / (u-d) + (uCd - dCu) / R(u-d) = \{(R - d)Cu + (u - R)Cd\} / \{R(u-d)\} \]

Application to Example

For Example Problem:

\[ R = 1 + Rf = 1.1 \quad (Rf \text{ assumed } = 10\% \text{ for simplicity}) \]
\[ Cu = \text{value of option in up state} = 15 \]
\[ Cd = \text{value of option in down state} = 0 \]
\[ u = \text{ratio of up movement of } S = 1.25 \]
\[ d = \text{ratio of down movement of } S = 0.8 \]

Portfolio Value = Option Price

\[ P = \frac{[(R - d)Cu + (u - R)Cd]}{R(u-d)} \]
\[ = \frac{[ (1.1 - 0.8) (15) + (1.25 - 1.1)(0)]}{1.1(1.25 - 0.8)} \]
\[ = \frac{[ 0.3(15)]}{1.1(.45)} = \frac{10}{1.1} = 9.09 \quad \text{as before} \]
Reformulation of Binomial Formulation

Option Price = \{(R - d)Cu + (u - R)Cd\} / \{R(u-d)\}

- We simplify writing of formula by substituting a single variable for a complex one:
  "q" ≡ \( \frac{R - d}{u - d} \)

- Option Price = \{(R - d)Cu + (u - R)Cd\} / \{R(u-d)\}
  = \( \frac{1}{R} \) \[q Cu + (1-q) Cd]\]

Option Value is weighted average of q, (1 – q)

q factor = risk-neutral “probability”

- Option Price = \( \frac{1}{R} \) \[q Cu + (1-q) Cd]\]

- This leads to an extraordinary interpretation!
  Value of option = “expected value” with binomial probabilities q and (1 - q)

- These called: “risk-neutral probabilities”

- Yet “q” defined by spread: q ≡ \( \frac{R - d}{u - d} \)
  actual probabilities do not enter into calculation!
Binomial Procedure using q

- “Arbitrage-enforced” pricing of options in binomial lattice proceeds as with “decision analysis” based valuation covered earlier

- Difference is that probabilities are no longer (p, 1 - p) but (q, 1 - q)

- From the perspective of calculation, (q, 1 - q) are exactly like probabilities
- However, never observed as frequencies, etc.
- Said to be “risk-neutral”, because derived from assumption of risk-free arbitrage

Summary

- Replicating Portfolio – key concept
  - A combination of asset and loan
  - Designed to give same outcomes as option
  - Leads to possibility of valuation of option

- Arbitrage Enforced Pricing
  - Option value determined by replicating portfolio
  - Provided that assumptions hold
  - “always” for traded assets;
  - Unclear for real options

- Risk Neutral “probabilities”
  - Represent Arbitrage-enforced valuation in lattice
Appendix

Another way to appreciate the Rationale for risk-free discount rate
In arbitrage enforced pricing of options

(with thanks to Michael Hanowsky)

Thought experiment

- We started with a call option on an asset that could end up with prices of 125 or 80
  - The Strike Price, K = 110
  - The option outcomes would be 15 or 0

- Suppose now a put on this asset with
  - Strike Price, K = 95
  - The option outcomes would be 0 or 15

- Now think of owning both together: the outcome is +15 whatever the asset price
Interpretation of Thought Case

- Both the Call and the Put are equivalent to a combination of the asset and borrowed money...

- The question for estimating the value of the options depends on the value assigned to \( r \) for the borrowed money...

- Since we can construct risk-free outcomes (as on previous slide)

  **Risk-free discount rate is appropriate!**