Black-Scholes Valuation

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Outline

- Background
- The Formula
  - Applicability ; Interpretation ; Intuition about form
  - Derivation principles
- Stochastic processes background
  - Random walk ; Wiener Process (Brownian motion)
  - Ito Process ; Geometric Brownian Motion
- Derivation background
- Applicability of this material to design

Meaning of “options analysis”

- Need to clarify the meaning of this term
- Methods presented for valuing options so far (lattice, etc) are all analyzing options. In that sense, they all constitute “options analysis”
- HOWEVER, in most literature “options analysis” means specific methods – based on replicating portfolios and random probability – epitomized by Black-Scholes

Keep this distinction in mind!

Background

- Development of “Options Analysis” Recent
- Depends on insights, solutions of
  - Black and Scholes ; Merton
  - Cox, Ross, Rubinstein
- This work has had tremendous impact
  - Development of huge markets for financial options, options “on” products (example: electric power)
- Presentation on options needs to discuss this – although much not applicable to engineering systems design

Key Papers and Events

- **Foundation papers:**
  - Black and Scholes (1973) "The Pricing of Options and Corporate Liabilities," J. of Political Economy, Vol. 81, pp. 637 - 654
  - Merton (1973) "Theory of Rational Option Pricing," Bell J. of Econ. and Mgt. Sci., Vol. 4, pp. 141 - 183
  - Cox, Ross and Rubinstein (1979) "Option Pricing: a Simplified Approach," J. of Financial Econ., Vol. 7, pp. 229-263. [The lattice valuation]
- **Events**
  - "Real Options" MIT Prof Myers ~ 1990
  - Nobel Prize in 1997 to Merton and Scholes (Black had died and was no longer eligible)

Black-Scholes Options Pricing Formula

$$C = S * N(d_1) - [K * e^{-rt} * N(d_2)]$$

It applies in a very special situation:
— a *European* call
— on a *non-dividend paying* asset

"European" ≡ only usable on a specific date
"American" ≡ usable any time in a period
(usual situation for options "in" systems)

"no dividends" -- so asset does not change over period

Black-Scholes Formula -- Terms

$$C = S * N(d_1) - [K * e^{-rt} * N(d_2)]$$

S , K = current price, strike price of asset
r = Rf = risk- free rate of interest
t = time to expiration
σ = standard deviation of returns on asset
N(x) = cumulative pdf up to x of normal distribution with average = 0, standard deviation = 1

$d_1 = [Ln (S/K) + (r + 0.5 \sigma^2) t] / (\sigma \sqrt{t})$
$d_2 = d_1 - (\sigma \sqrt{t})$

Black-Scholes Formula -- Intuition

$$C = S * N(d_1) - [K * e^{-rt}] * N(d_2)$$

Note that, since N(x) < 1.0, the B-S formula expresses option value, C, as
- a fraction of the asset price, S, less
- a fraction of discounted amount, $(K * e^{-rt})$

These are elements needed to create a replicating portfolio (see "Arbitrage-enforced pricing" slides). Indeed, B-S embodies this principle with a continuous pdf.

Black-S Formula – Derivation Principles

- Formula is a solution to a "Stochastic Differential Equation" (or SDE) that defines movement of value of option over time
- SDE's defined by Ito from Japan
  – the general form known as an "Ito Process"
- Specific form of equation solved
  – embodies principle of replicating portfolio
  – makes specific assumptions about nature of movement value in a competitive market
- These ideas discussed next

Random Walks

- A "Standardized Normal Random variable", e(t)
  – It is a "Normal" distribution (bell-shaped)
  – Mean ≡ 0 ; Standard deviation ≡ 1
- A "random walk" is a process defined by
  – $z(t + 1) = z(t) + e(t) (\Delta t)^{0.5}$
- Difference between 2 periods: $z(t_k) – z(t_j)$
  – Expected value = 0 ; Variance = tk – tj
  – Differences for non-overlapping periods are uncorrelated
- This is a random process

Wiener Process

- This is result of "random walk" as $\Delta t \rightarrow 0$
- Formally: $z(t + 1) = z(t) + e(t) (\Delta t)^{0.5}$
- Becomes: $dz = e(t) (\Delta t)^{0.5}$
- Also known as "Brownian Motion" in science or "white noise" in engineering
- As for random walk:
  - $z(t) – z(s)$ is a normal random variable
  - for any 4 times $t1 < t2 < t3 < t4$ $[z(t1) - z(t2)]$ and $[z(t3) - z(t4)]$ are uncorrelated

Generalized Wiener process

- An extension of Brownian motion

$$dx(t) = a\,dt + b\,dz$$

- In short, it
  - represents a growth trend: $a\,dt$
  - Plus white noise: $b\,dz$
- It can be solved: $x(t) = x(0) + a\,t + b\,z(t)$
- This is similar to what lattice represents – but see next slides

Ito Process

- A further extension…
- Basic Eqn: $dx(t) = a\,dt + b\,dz$
- Becomes: $dx(t) = a(x, t)\,dt + b(x, t)\,dz$
- In short, coefficients can change with time
- This is a "stochastic differential equation"
  - Stochastic because it varies randomly with time

Application to Asset Prices -- GBM

- Asset prices assumed to fluctuate around a multiplicative growth trend
  - For example S_0 ➔ u (S_0) or d (S_0)
- The continuous version of this is:

$$d[\ln S(t)] = v\,dt + \sigma\,dz$$

- This is a generalized Wiener process
- With solution: $\ln S(t) = \ln S_0 + vt + \sigma z(t)$
- This is: Geometric Brownian Motion (GBM)

Standard Ito form

- This is the solution for S(t)….
- $d\,S(t) / S(t) = (v + 0.5\,\sigma^2)\,dt + \sigma\,dz$
  - Solution not obvious -- A special case of Ito's lemma
- Interpret this as saying that:
  Relative change of asset value, $d\,S(t) / S(t)$
  = a trend (constant) dt
  + random factor scaled by σ
- Alternatively: $d\,S(t) = \mu\,S\,dt + \sigma\,S\,dz$
- where $\mu = v + 0.5\,\sigma^2$ [$0.5\,\sigma^2$ is correction factor]

Ito's lemma

- If: x(t) is defined by Ito process
  $dx(t) = a(x, t)\,dt + b(x, t)\,dz$
- And y(t) = F(x,t) some function (or "derivative" or specifically an option)
- Then:
  $d\,y(t) = [(\delta F/\delta x)a + \delta F/\delta t + 0.5(\delta^2 F/ \delta x^2)b^2]dt + (\delta F/\delta x)\,b\,dz$
- In words: given a "derivative" of an asset, F(x,t), we have an equation defining value of derivative

Derivation Background

- Suppose value of Asset is random process:
  $$d\,S(t) = \mu\, S\, dt + \sigma\, S\, dz$$
- And that we can borrow money at rate r
- The price of a derivative (an option) f(S,t) of this asset satisfies the Black-Scholes equation:
  $$\delta f/\delta t + (\delta f/\delta S)\, r\, S + (\delta^2 f/\delta S^2)\, \sigma^2\, S^2 = rf$$
- Unless this property is met – arbitrage opportunity exists
- Solution to equation defines price of derivative

Black-Scholes Formula as Solution

- It is the solution to the Black-Scholes equation
- Meeting the boundary conditions:
  - It is a call
  - There is only 1 exercise time (European option)
  - The asset "pays no dividends" – that is, gives off no intermediate benefit (mines or oil wells generate 'dividends' in exploitation, so B-S does not apply)
- Development a brilliant piece of work
- Why do we care?

Why does this matter?

- Development of Formula showed the way for financial analysts
- Essentially "no" other significant closed form solutions…
- But solutions worked out numerically through lattice (and more sophisticated) analyses
- Led to immense development of use of all kinds of "derivatives"   (an alternative jargon word that refers to various options)

Why does this matter TO US?

- What does B-S mean to designers of technological systems?
- Important to understand the assumptions behind Black-Scholes equation and approach
- Extent these assumptions are applicable to us, determines the applicability of the approach
- Much research needed to
  - address this issue
  - Develop alternative approaches to valuing flexibility

Price Assumption

- B-S approach assumes Asset has a "price"
- When is this true?
  - System produces a commodity (oil, copper) that has quoted prices set by world market
- When this may be true
  - System produces goods (cars, CDs) that lead to revenues and thus value – HOWEVER, product prices depend on both design and management decisions
- When this is not true
  - System delivers services that are not marketable, for example, national defense…

Replicating Portfolio Assumption

- B-S analysis assumes that it is possible to set up replicating portfolio for the asset
- When is this true
  - Product is a commodity
- When this might assumed to be true
  - Even if market does not exist, we might assume that a reasonable approximation might be constructed (using shares in company instead of product price)
- When this is probably a stretch too far
  - Private concern, owners unconcerned with arbitrage against them, who may want to use actual probabilities…

Volatility Assumption

- B-S approach assumes that we can determine volatility of asset price
- When this is true
  - There is an established market with a long history of trades that generates good statistics
- When this is questionable
  - The market is not observable (for example, because data are privately held or negotiated)
  - Assets are unique (a prestige or special purpose building or special location)
- When this is not true
  - New technology or enterprise with no data

Duration Assumption

- B-S approach assumes volatility of asset price is stable over duration of option
- When this is true
  - Short-term options (3 months, a year?) in a stable industry or activity
- When this is questionable
  - Industries that are in transition – technologically, in structure, in regulation – such as communications
- When this is not true
  - Long-duration projects in which – major changes in states of markets, regulations or technologies are highly uncertain (Exactly where we want flexibility!)

Take-Away from this Discussion

- In many situations the basic premises of "options analysis" – as understood in finance – are unlikely to apply to the design and management of engineering systems
- Yet these systems, typically being long-life, are likely to be especially uncertain – and thus most in need of flexibility – of "real options"
- We thus need to develop pragmatic ways to value options for engineering systems
  TOPIC OF SUBSEQUENT PRESENTATIONS

Summary

- Black-Scholes formula elegant and historically most important
- Its derivation based on some fundamental developments in Stochastic processes
  - Random walk; Wiener and Ito Processes ; GBM
- Underlying assumptions limit use of approach
  - Price; Replicating Portfolio; Volatility; Duration
- Developing useful, effective approaches for design is an urgent, important task

Appendix

MEAN REVERSION PROCESSES

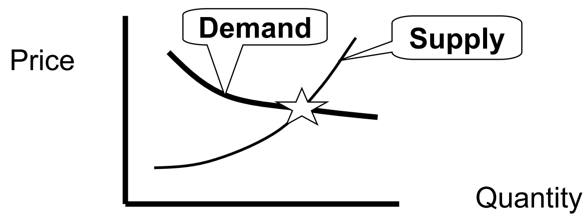
The Concept

- "Mean Reversion": the concept that a variable process wants to revert (= come back to) some natural level (which would be its long-run average value)

- Physical analogy: A spring (think of a shock absorber on a car) that
  - has an equilibrium position
  - Counteracts any displacement (stretch or squish)
  - With force proportional to displacement (stronger further away from mean)

Applicability

- Mean Reversion widely associated with commodities of all sorts (oil, copper, money)
- The economic rationale is that demand and supply should be in equilibrium



Limitations

- **Supply curve shifts over time**
  - "low hanging fruit" get picked – low cost oil fields or mines are exhausted – so shifts upward
  - short-run and long-run costs differ (it takes time to develop new sources), so shifts at extremes
- **Demand curve also shifts over time**
  - Economic booms (dot.com, China construction)
  - … and busts
  - Technology shifts (glass fibre replace copper wire)
- **So notion of equilibrium somewhat fuzzy…**

Practice

- Mean reverting models of stochastic processes widely used. Many believe them to be more realistic than binomial diffusion
- However, no definitive proof (which would in any case depend on product)
- Available commercially (e.g. Crystal Ball ®)
- They come in various versions, an example follows…

Arithmetic Mean Reversion

- The Generalized Brownian motion model is:
  d [ln S(t)] = v dt + σ dz (Slide 14)
- An arithmetic mean reversion process can be:
  d [ln S(t)] = v [m-x] dt + σ dz
  Where x = ln S
- Follow up reading:
  Schwartz (19970 "The stochastic behavior of commodity prices: implications for valuation and hedging," J. of Finance 52(3), pp 923-973

  Also: http:// www.puc-rio.br/marco.ind/revers. html