Black-Scholes Valuation

Richard de Neufville
Professor of Engineering Systems and of Civil and Environmental Engineering
MIT

Outline

• Background

• The Formula
  – Applicability; Interpretation; Intuition about form
  – Derivation principles

• Stochastic processes background
  – Random walk; Wiener Process (Brownian motion)
  – Ito Process; Geometric Brownian Motion

• Derivation background

• Applicability of this material to design
Meaning of “options analysis”

- Need to clarify the meaning of this term
- Methods presented for valuing options so far (lattice, etc) are all analyzing options. In that sense, they all constitute “options analysis”
- HOWEVER, in most literature “options analysis” means specific methods – based on replicating portfolios and random probability – epitomized by Black-Scholes

Keep this distinction in mind!

Background

- Development of “Options Analysis” Recent
- Depends on insights, solutions of
  - Black and Scholes; Merton
  - Cox, Ross, Rubinstein
- This work has had tremendous impact
  - Development of huge markets for financial options, options “on” products (example: electric power)
- Presentation on options needs to discuss this – although much not applicable to engineering systems design
Key Papers and Events

- **Foundation papers:**

- **Events**
  - “Real Options” MIT Prof Myers ~ 1990
  - Nobel Prize in 1997 to Merton and Scholes (Black had died and was no longer eligible)

---

Black-Scholes Options Pricing Formula

\[ C = S \times N(d_1) - [K \times e^{-rt} \times N(d_2)] \]

It applies in a very special situation:
- a **European call**
- on a **non-dividend paying asset**

“European” ≡ only usable on a specific date
“American” ≡ usable any time in a period (usual situation for options “in” systems)

“no dividends” -- so asset does not change over period
Black-Scholes Formula -- Terms

\[ C = S \times N(d_1) - [K \times e^{-rt} \times N(d_2)] \]

- \( S \), \( K \) = current price, strike price of asset
- \( r \) = risk-free rate of interest
- \( t \) = time to expiration
- \( \sigma \) = standard deviation of returns on asset
- \( N(x) \) = cumulative pdf up to \( x \) of normal distribution with average = 0, standard deviation = 1

\[
\begin{align*}
    d_1 &= \frac{\ln(S/K) + (r + 0.5 \sigma^2)t}{\sigma \sqrt{t}} \\
    d_2 &= d_1 - (\sigma \sqrt{t})
\end{align*}
\]

Black-Scholes Formula -- Intuition

\[ C = S \times N(d_1) - [K \times e^{-rt} \times N(d_2)] \]

Note that, since \( N(x) < 1.0 \), the B-S formula expresses option value, \( C \), as
- a fraction of the asset price, \( S \), less
- a fraction of discounted amount, \( (K \times e^{-rt}) \)

These are elements needed to create a replicating portfolio (see “Arbitrage-enforced pricing” slides). Indeed, B-S embodies this principle with a continuous pdf.
Black-S Formula – Derivation Principles

- Formula is a solution to a “Stochastic Differential Equation” (or SDE) that defines movement of value of option over time

- SDE’s defined by Ito from Japan
  - the general form known as an “Ito Process”

- Specific form of equation solved
  - embodies principle of replicating portfolio
  - makes specific assumptions about nature of movement value in a competitive market

- These ideas discussed next

Random Walks

- A “Standardized Normal Random variable”, e(t)
  - It is a “Normal” distribution (bell-shaped)
  - Mean ≡ 0 ; Standard deviation ≡ 1

- A “random walk” is a process defined by
  - \( z(t + 1) = z(t) + e(t) (\Delta t)^{0.5} \)

- Difference between 2 periods: \( z(t_k) - z(t_j) \)
  - Expected value = 0 ; Variance = \( t_k - t_j \)
  - Differences for non-overlapping periods are uncorrelated

- This is a random process
**Wiener Process**

- This is result of “random walk” as Δt ➞ 0
- Formally: \( z(t + 1) = z(t) + e(t) (Δt)^{0.5} \)
- Becomes: \( dz = e(t) (Δt)^{0.5} \)

- Also known as “Brownian Motion” in science or “white noise” in engineering

- As for random walk:
  - \( z(t) - z(s) \) is a normal random variable
  - for any 4 times \( t_1 < t_2 < t_3 < t_4 \) \[z(t_1) - z(t_2]\] and \[z(t_3) - z(t_4)\] are uncorrelated

---

**Generalized Wiener process**

- An extension of Brownian motion
  \( dx(t) = a \ dt + b \ dz \)

- In short, it
  - represents a growth trend: \( a \ dt \)
  - Plus white noise: \( b \ dz \)

- It can be solved: \( x(t) = x(0) + a \ t + b \ z(t) \)

- This is similar to what lattice represents – but see next slides
Ito Process

- A further extension...

  - Basic Eqn: \( dx(t) = a \, dt + b \, dz \)
  - Becomes: \( dx(t) = a(x, t) \, dt + b(x, t) \, dz \)

- In short, coefficients can change with time

- This is a “stochastic differential equation”
  - Stochastic because it varies randomly with time

Application to Asset Prices -- GBM

- Asset prices assumed to fluctuate around a multiplicative growth trend
  - For example \( S_0 \rightarrow u \, S_0 \) or \( d \, S_0 \)

- The continuous version of this is:
  \[ d \, [\ln S(t)] = \nu \, dt + \sigma \, dz \]
  - This is a generalized Wiener process

- With solution: \( \ln S(t) = \ln S_0 + vt + \sigma \, z(t) \)

- This is: Geometric Brownian Motion (GBM)
**Standard Ito form**

- This is the solution for \( S(t) \)....

- \[
    \frac{d S(t)}{S(t)} = (v + 0.5 \sigma^2) \, dt + \sigma \, dz
\]
  - Solution not obvious -- A special case of Ito’s lemma

- Interpret this as saying that:
  Relative change of asset value, \( \frac{d S(t)}{S(t)} \)
  = a trend (constant) \( dt \)
  + random factor scaled by \( \sigma \)

- Alternatively: \[
    d S(t) = \mu S \, dt + \sigma S \, dz
\]
- where \( \mu = v + 0.5 \sigma^2 \) [0.5 \( \sigma^2 \) is correction factor]

---

**Ito’s lemma**

- If: \( x(t) \) is defined by Ito process
  \[
  dx(t) = a(x, t) \, dt + b(x, t) \, dz
  \]

- And \( y(t) = F(x, t) \) some function (or “derivative” or specifically an option)

- Then:
  \[
  d y(t) = \left\{ (\delta F/\delta x) a + \delta F/\delta t + 0.5(\delta^2 F/ \delta x^2)b^2 \right\} dt
  + (\delta F/\delta x) b \, dz
  \]

- In words: given a “derivative” of an asset, \( F(x, t) \),
  we have an equation defining value of derivative
Derivation Background

- Suppose value of Asset is random process:
  \[ d S(t) = \mu S \, dt + \sigma S \, dz \]
- And that we can borrow money at rate \( r \)
- The price of a derivative (an option) \( f(S,t) \) of this asset satisfies the Black-Scholes equation:
  \[ \frac{\partial f}{\partial t} + (\frac{\partial f}{\partial S}) r \, S + (\frac{\partial^2 f}{\partial S^2}) \sigma^2 S^2 = rf \]
- Unless this property is met – arbitrage opportunity exists
- Solution to equation defines price of derivative

Black-Scholes Formula as Solution

- It is the solution to the Black-Scholes equation
- Meeting the boundary conditions:
  - It is a call
  - There is only 1 exercise time (European option)
  - The asset “pays no dividends” – that is, gives off no intermediate benefit (mines or oil wells generate ‘dividends’ in exploitation, so B-S does not apply)
- Development a brilliant piece of work
- Why do we care?
Why does this matter?

- Development of Formula showed the way for financial analysts

- Essentially “no” other significant closed form solutions...

- But solutions worked out numerically through lattice (and more sophisticated) analyses

- Led to immense development of use of all kinds of “derivatives” (an alternative jargon word that refers to various options)

Why does this matter TO US?

- What does B-S mean to designers of technological systems?

- Important to understand the assumptions behind Black-Scholes equation and approach

- Extent these assumptions are applicable to us, determines the applicability of the approach

- Much research needed to
  - address this issue
  - Develop alternative approaches to valuing flexibility
Price Assumption

- B-S approach assumes Asset has a “price”

- When is this true?
  - System produces a commodity (oil, copper) that has quoted prices set by world market

- When this may be true
  - System produces goods (cars, CDs) that lead to revenues and thus value – HOWEVER, product prices depend on both design and management decisions

- When this is not true
  - System delivers services that are not marketable, for example, national defense...

Replicating Portfolio Assumption

- B-S analysis assumes that it is possible to set up replicating portfolio for the asset

- When is this true
  - Product is a commodity

- When this might assumed to be true
  - Even if market does not exist, we might assume that a reasonable approximation might be constructed (using shares in company instead of product price)

- When this is probably a stretch too far
  - Private concern, owners unconcerned with arbitrage against them, who may want to use actual probabilities...
Volatility Assumption

- B-S approach assumes that we can determine volatility of asset price

- When this is true
  - There is an established market with a long history of trades that generates good statistics

- When this is questionable
  - The market is not observable (for example, because data are privately held or negotiated)
  - Assets are unique (a prestige or special purpose building or special location)

- When this is not true
  - New technology or enterprise with no data

Duration Assumption

- B-S approach assumes volatility of asset price is stable over duration of option

- When this is true
  - Short-term options (3 months, a year?) in a stable industry or activity

- When this is questionable
  - Industries that are in transition – technologically, in structure, in regulation – such as communications

- When this is not true
  - Long-duration projects in which – major changes in states of markets, regulations or technologies are highly uncertain (Exactly where we want flexibility!)
Take-Away from this Discussion

- In many situations the basic premises of “options analysis” – as understood in finance – are unlikely to apply to the design and management of engineering systems

- Yet these systems, typically being long-life, are likely to be especially uncertain – and thus most in need of flexibility – of “real options”

- We thus need to develop pragmatic ways to value options for engineering systems

TOPIC OF SUBSEQUENT PRESENTATIONS

Summary

- Black-Scholes formula elegant and historically most important

- Its derivation based on some fundamental developments in Stochastic processes
  - Random walk; Wiener and Ito Processes; GBM

- Underlying assumptions limit use of approach
  - Price; Replicating Portfolio; Volatility; Duration

- Developing useful, effective approaches for design is an urgent, important task
Appendix

MEAN REVERSION PROCESSES

The Concept

- “Mean Reversion”: the concept that a variable process wants to revert (= come back to) some natural level (which would be its long-run average value)

- Physical analogy: A spring (think of a shock absorber on a car) that
  - has an equilibrium position
  - Counteracts any displacement (stretch or squish)
  - With force proportional to displacement (stronger further away from mean)
Applicability

- Mean Reversion widely associated with commodities of all sorts (oil, copper, money)

- The economic rationale is that demand and supply should be in equilibrium

Limitations

- Supply curve shifts over time
  - “low hanging fruit” get picked – low cost oil fields or mines are exhausted – so shifts upward
  - short-run and long-run costs differ (it takes time to develop new sources), so shifts at extremes

- Demand curve also shifts over time
  - Economic booms (dot.com, China construction)
  - ... and busts
  - Technology shifts (glass fibre replace copper wire)

- So notion of equilibrium somewhat fuzzy...
Practice

- Mean reverting models of stochastic processes widely used. Many believe them to be more realistic than binomial diffusion

- However, no definitive proof (which would in any case depend on product)

- Available commercially (e.g. Crystal Ball ®)

- They come in various versions, an example follows…

Arithmetic Mean Reversion

- The Generalized Brownian motion model is:
  \[ d \ln S(t) = \nu \, dt + \sigma \, dz \]  
  (Slide 14)

- An arithmetic mean reversion process can be:
  \[ d \ln S(t) = \nu \left[ m - x \right] \, dt + \sigma \, dz \]

Where \( x = \ln S \)

- Follow up reading:

  Also: http://www.puc-rio.br/marco.ind/revers. html